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On linear operators and bases on Köthe spaces

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Abstract

We make a survey of results published by the authors about the backward and forward unilateral weighted shift operators in Kóthe spaces, the so-called generalized derivation and integration operators, extending well-known results for spaces of analytic functions.

Keywords: Shift operators; Köthe spaces.
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1 Introduction

Weighted shift operators have been studied by many authors in different contexts, for instance the work by N. K. Nikol'skiĭ in the spaces spaces ℓ^p, [15–18], R. Gellar in Banach spaces, [2–4] and Grabiner in Banach algebras of power series spaces, [5–7].

Forward weighted operators (multiplication and integration operators) play a remarkable role in the study of bases in spaces of analytic functions and have been considered by many Russian mathematicians, [11]. The Gončarov polynomials, that under certain conditions are a basis in analytic spaces [1, 9], are related to the backward weighted operator (derivation operator).

We work with Köthe spaces and weighted shifts on them (generalized integration and derivation operators). We characterize the forward shift-invariant isomorphisms and then determine some some quasi-power bases. Our results include, as particular cases, those of Nagnibida for the multiplication and integration operators on the space of analytic functions on a disc, [11] and Prada for the multiplication operator on infinite power series spaces, [19, 20]. Using the backward shift operator we get conditions for the Gončarov polynomials to be a basis.

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2 Basic results

Denote by $\lambda^p(A)$, $1 \le p < \infty$, the Köthe (echelon) space given by the matrix $A = (a_n^k)_{n=0}^{\infty}$, $0 < a_n^k \le a_n^{k+1}$ for all n,k, that is

$$\lambda^p(A) = \left\{ x = (x_n)_{n=0}^{\infty}, x_n \in \mathbb{C} : \sum_{n=0}^{\infty} \left(|x_n|\, a_n^k \right)^p < \infty, \forall k = 0,1,2,\dots \right\}.$$

$\lambda^p(A)$ is a Frèchet space, [14], with the norms

$$\|x\|_k = \left(\sum_{n=0}^{\infty} \left(|x_n|\, a_n^k \right)^p \right)^{1/p}, \quad k = 0,1,2,\dots$$

When $p = 0, \infty$, we have

$$\lambda^0(A) = \left\{ x = (x_n), x_n \in \mathbb{C} : \|x\|_k = \sup \left(|x_n|\, a_n^k \right) < \infty, \ \forall k = 0,1,2,\dots \right\}.$$
$$\lambda^\infty(A) = \left\{ x = (x_n), x_n \in \mathbb{C} : \|x\|_k = \lim \left(|x_n|\, a_n^k \right) = 0, \ \forall k = 0,1,2,\dots \right\}.$$

The canonical basis in the spaces $\lambda^p(A)$, $p = 0$, $1 \le p < \infty$, is denoted by $\delta_n = (\delta_{n,k})_{k=0}^{\infty}$, where $\delta_{n,k}$ is the Kronecker delta.

The dual space of $\lambda^p(A)$, $1 \le p < \infty$, $p = 0$, $\frac{1}{p} + \frac{1}{q} = 1$ is given by

$$(\lambda^p(A))^\times = \left\{ (x_n)_{n=0}^{\infty}, x_n \in \mathbb{C} : \left(\sum_{n=0}^{\infty} \frac{|x_n|^q}{(a_n^k)^q} \right)^{\frac{1}{q}} < \infty, \text{ for a suitable } k \right\}, \ 1 < p < \infty.$$

$$\left(\lambda^1(A)\right)^\times = \left\{ (x_n)_{n=0}^{\infty}, x_n \in \mathbb{C} : \sup_{n \ge 0} \left\{ \frac{|x_n|}{a_n^k} \right\} < \infty, \text{ for a suitable } k \right\}, \ p = 1.$$

$$\left(\lambda^0(A)\right)^\times = \left\{ (x_n)_{n=0}^{\infty}, x_n \in \mathbb{C} : \sum_{n=0}^{\infty} \frac{|x_n|}{a_n^k} < \infty, \text{ for a suitable } k \right\}, \ p = 0.$$

Recall that the coordinate operators are continuous, [14].

$\lambda^p(A)$, $p \in [1,\infty)$, $p = 0$ is the projective limit of the Banach spaces $\ell^p(a^k)$, $c_0(a^k)$, diagonal transformations of ℓ^p, c_0, with the usual topology:

$$\ell^p(a^k) = \left\{ x = (x_n)_{n=0}^{\infty} : (x_n\, a_n^k)_{n=0}^{\infty} \in \ell^p \right\}, \quad 1 \le p < \infty$$
$$c_0(a^k) = \left\{ x = (x_n)_{n=0}^{\infty} : (x_n\, a_n^k)_{n=0}^{\infty} \in c_0 \right\}.$$

The space $\ell^1(a^k)$ is a Banach algebra if and only if the following condition holds

$$\exists C(k) > 0 : a_{m+n}^k \le C(k) a_m^k a_n^k, \quad \forall n,m = 0,1,2,\dots$$

The space $\lambda^1(A)$ is nuclear if and only if

$$\forall k, \exists r(k) : \frac{a_n^k}{a_n^{r(k)}} \in \ell^1$$

and therefore $\lambda^p(A) = \lambda^1(A) = \lambda^0(A)$, $1 \leq p < \infty$ [14].

If $\lambda = (\lambda_n)$ is a sequence of nonzero complex numbers with $\lambda_0 = 1$ to simplify computations, the operator J_λ defined by

$$J_\lambda(\delta_n) = \frac{\lambda_{n+1}}{\lambda_n}\delta_{n+1}$$

is called the generalized integration operator. If $\lambda_n = \frac{1}{n!}$, $J_\lambda = J$, is the integration operator and if $\lambda_n = 1$, $J_\lambda = U$, is the multiplication one (shift operator), see [11].

We assume that the operator J_λ, where the sequence (λ_n) are positive real numbers without lost of generality, is continuous on $\lambda^p(A)$, that is the following condition is fulfilled

$$\forall k, \exists r = r(k) : \sup_{n\geq 0}\left(\frac{\lambda_{n+1}}{\lambda_n}\frac{a_n^{k+1}}{a_n^k}\right) < \infty$$

If (d_n) is a sequence of positive real numbers, the operator D given by

$$D(\delta_n) = \frac{d_{n-1}}{d_n}\delta_{n-1}$$

is called the generalized derivation operator, being the usual derivation when $d_n = \frac{1}{n!}$.

3 Isomorphisms commuting with J_λ. Bases in Köthe spaces

We characterize the isomorphisms between Köthe spaces that commute with the generalized integration operator J_λ determining some bases, related with it, on $\lambda^1(A)$.

Theorem 1. *Let $T\colon \lambda^1(A) \to \lambda^1(A)$ be a continuous linear operator. $\left\{\frac{1}{\lambda_n}T^n x\right\}_{n=0}^{\infty}$, $x \in \lambda^1(A)$ is a basis in $\lambda^1(A)$ if and only if there exists an isomorphism $S\colon \lambda^1(A) \to \lambda^1(A)$ such that $T \circ S = S \circ J_\lambda$ and $x = S\delta_0$.*

Proof. If $\left\{\frac{1}{\lambda_n}T^n x\right\}$, $n \geq 0$, is a basis in $\lambda^1(A)$, then there exists an isomorphism S such that $S\delta_n = \frac{1}{\lambda_n}T^n x$, $n = 0, 1, 2, \ldots$ It follows that $S\delta_0 = x$ and for $n \in \mathbb{N}$

$$(S \circ J_\lambda)\delta_n = \frac{\lambda_{n+1}}{\lambda_n}S\delta_{n+1} = \frac{1}{\lambda_n}(T \circ T^n x) = (T \circ S)\delta_n.$$

Conversely

$$T^n x = (T^{n-1}TS)\delta_0 = (T^{n-1}SJ_\lambda)\delta_0 = (SJ_\lambda^n)\delta_0 = \lambda_n S\delta_n.$$

Corollary 2. $\left\{\frac{1}{\lambda_n}J_\lambda^n x\right\}$, *$x \in \lambda^1(A)$ is a basis in $\lambda^1(A)$ if and only if there exists an isomorphism $T\colon \lambda^1(A) \to \lambda^1(A)$ that commutes with J_λ and $x = T\delta_0$.*

Proposition 3. *[13] A linear operator $T\colon \lambda^1(A) \to \lambda^1(A)$ is continuous and commutes with J_λ if and only if*

$$T = \sum_{m=0}^{\infty}\frac{b_m}{\lambda_m}J_\lambda^m, \quad b = (b_m)_{m=0}^{\infty} = T\delta_0$$

and the condition

$$\forall k, \exists r = r(k) : \sup_n\left\{\sum_{m=0}^{\infty}|b_m|\frac{\lambda_{m+n}}{\lambda_m\lambda_n}\frac{a_{m+n}^k}{a_n^r}\right\} < \infty$$

is fulfilled.

Proposition 4 (c.f. [13]). *Let T be a linear operator from $\lambda^1(A)$ onto itself commuting with J_λ and $b = (b_n) = T(\delta_0)$ ($b_0 \neq 0$). If T^{-1} is the formal operator given by the inverse matrix of T, $c = (c_n) = T^{-1}(\delta_0)$ and k,*

$$\forall k,\ \exists r = r(k) : \sup_{m\geq 0, n\geq 0} \left\{ \frac{\lambda_{m+n}}{\lambda_m \lambda_n} \frac{a^k_{m+n}}{a^k_m a^r_n} \right\} < \infty,$$

then T is an isomorphism if and only if $b, c \in \lambda^1(A)$.

Remark 1. Recall that the matrix $(t_{i,j})$ of a continuous linear operator T commuting with J_λ is lower triangular so, formally, $(t_{i,j})$ has an inverse of the same type if $T\delta_0 = (b_n)$ with $b_0 \neq 0$. The operator T^{-1} given by this inverse matrix is always linear and commutes with J_λ. Then a continuous operator T is an isomorphism if and only if T^{-1} is continuous and T^{-1} can be written

$$T^{-1} = \sum_{n=0}^{\infty} \frac{c_n}{\lambda_n} J^n_\lambda, \quad c = (c_n) = T^{-1}(\delta_0).$$

Theorem 5 (c.f. [13]). *Assume the following conditions:*

1. *$a^k_{m+n} \leq C_k a^k_m a^k_n$, $\forall k$, that is, the spaces $\ell^1(a^k)$ are Banach algebras.*
2. *$\lambda_{m+n} \leq C\lambda_m \lambda_n$, $\forall m, n$.*
3. *Let $T = \sum_{n=0}^{\infty} \frac{b_n}{\lambda_n} J^n_\lambda$ be a linear operator on $\lambda^1(A)$ commuting with J_λ.*

Then T is an isomorphism if and only if any of the following equivalent conditions are satisfied:

1. *The sequence $(\frac{b_n}{\lambda_n})$ is an exponential (invertible) element of all the Banach algebras $\ell^1(b^k)$, $b^k_n = \lambda_n a^k_n$, for all k.*
2. *The sequence $(b_n) \in \lambda^1(A)$ and*

$$\sum_{n=0}^{\infty} \frac{b_n}{\lambda_n} z^n \neq 0, \quad |z| \leq r_k, \ \ r_k = \lim_n (\lambda_n a^k_n)^{\frac{1}{n}} \ \textit{for all } k.$$

Corollary 6 (c.f. [13]). *Let T be a linear operator commuting with J_λ*

$$T = \sum_{n=0}^{\infty} \frac{b_n}{\lambda_n} J^n_\lambda, \, b = (b_n) = T\delta_0, \, b_0 \neq 0.$$

Suppose the following conditions are satisfied:

$$\begin{aligned} \forall k,\ \exists C_k > 0 :\ & a^k_{m+n} \leq C_k a^k_m a^k_n, \\ & \lambda_{m+n} \leq C\lambda_m \lambda_n, \forall m, n. \end{aligned}$$

If $\left(\frac{b_n}{\lambda_n}\right)$ is an exponential (invertible) element of $\lambda^1(B)$, $B = \left(b^k_n\right) = (\lambda_n a^k_n)$, then the system

$$\left\{ \lambda_n T^n \left(\frac{b_j}{\lambda_j} \right)_{j=0}^{\infty} \right\}_{n=0}^{\infty}$$

is a basis in $\lambda^1(A)$.

Proposition 7. *[13] Let T be a linear operator on $\lambda^1(A)$ commuting with J_λ, $T = \sum_{n=0}^{\infty} \frac{b_n}{\lambda_n} J^n_\lambda$, $b_0 \neq 0$.*

1. *If there exists* $M_k = \lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n}\frac{a^k_{n+1}}{a^k_n}$, $M_k \neq 0$, *for a suitable* k, *then the function* $\phi(z) = \sum_{n=0}^{\infty} \frac{b_n}{\lambda_n} z^n$ *is an holomorphic one with no zeros in a disc* $D(0,\rho)$, *with* $\rho \geq M_k$.

2. *If* $\lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n}\frac{a^k_{n+1}}{a^k_n} = \infty$ *for a suitable* k, *then the function* $\phi(z) = \sum_{n=0}^{\infty} \frac{b_n}{\lambda_n} z^n$ *is an entire function without zeros.*

Proposition 8 (c.f. [13])**.** *Let* T *be a linear operator on* $\lambda^1(A)$ *commuting with* J_λ

$$T = \sum_{n=0}^{\infty} \frac{b_n}{\lambda_n} J^n_\lambda,\ b_0 \neq 0.$$

Suppose that

$$\forall k, \exists M_k = \sup_n \left\{ \frac{\lambda_{n+1}}{\lambda_n}\frac{a^k_{n+1}}{a^k_n} \right\} < \infty.$$

If the function $\phi(z) = \sum_{n=0}^{\infty} \frac{b_n}{\lambda_n} z^n$ *is holomorphic without zeros in a disc* $\mathbb{D}_\rho$, $\rho > \sup_k \{M_k\}$ *or* $\rho = \infty$, *then* T *is an isomorphism from* $\lambda^1(A)$ *onto itself.*

Proposition 9 (c.f. [13])**.** *If for a suitable* k,

$$\lim_{n\to\infty} \lambda_n a^k_n = \infty, \quad \lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n}\frac{a^k_{n+1}}{a^k_n} = \infty, \quad \limsup_{n\to\infty} \frac{\log(n+1)}{\log(\frac{\lambda_{n+1}}{\lambda_n}\frac{a^k_{n+1}}{a^k_n})} = 0,$$

then the only entire functions without zeros that give continuous linear operators on $\lambda^1(A)$ *are the constants.*

Example 10. *The space of holomorphic functions,* $\mathscr{H}(\mathbb{D}_R)$, *on the disc* $\mathbb{D}_R = \mathbb{D}(0,R)$, $0 < R \leq \infty$ *is a Köthe space* $\lambda^1(A)$, *with* $A = (a^k_n) = (t^n_k)$, *where* (t_k) *is an increasing sequence of real positive numbers converging to* R.

- *If* $\lambda_n = 1$, $\forall n$ *then a continuous linear operator* $T = \sum_{n=0}^{\infty} b_n U^n$ *on* $\mathscr{H}(\mathbb{D}_R)$, *commuting with the multiplication operator* U, *is an isomorphism if and only if the function* $\phi(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathscr{H}(\mathbb{D}_R)$ *and has no zeros in the disc* $\mathbb{D}_R$, *see [11].*
- *If* $\lambda_n = \frac{1}{n!}$, $\forall n$ *then* $J_\lambda = J$ *and a linear continuous operator* T *on* $\mathscr{H}(\mathbb{D}_R)$, *commuting with* J, *is an isomorphism if and only if the function* $\phi(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathscr{H}(\mathbb{D}_R)$ *and* $b_0 \neq 0$, *see [11].*

Example 11. *The space* $\lambda^1(A) = \Lambda_\infty(\alpha)$, $A = (e^{k\alpha_n})$ *with* (α_n) *an increasing sequence of positive numbers going to infinity, is an infinite power series space.*

- *If* $\lambda_n = 1$, $\forall n$, *and* $\alpha_{m+n} \leq C + \alpha_n + \alpha_m$, $\forall m,n$, *then a continuous linear operator* T *on* $\Lambda_\infty(\alpha)$, *commuting with* U, *is an isomorphismif and only if the sequence* $T\delta_0 = (b_n) \in \Lambda_\infty(\alpha)$ *and the function* $\phi(z) = \sum_{n=0}^{\infty} b_n z^n$ *has no zeros in the closed disk* $D(0,1)$ *(if* $\lim_{n\to\infty} \frac{\alpha_n}{n} = 0$*) or has no zeros in the complex plane (if* $\lim_{n\to\infty} \frac{\alpha_n}{n} > 0$*) [20].*
- *If* $\lambda_n = \frac{1}{n!}$, $\forall n$, $\alpha_{m+n} \leq C + \alpha_n + \alpha_m$, $\forall m,n$, *and* $\lim_{n\to\infty} \frac{\alpha_n}{n} < \infty$, *then a continuous linear operator* T *is an isomorphism on* $\Lambda_\infty(\alpha)$ *commuting with* J *if and only if* $T\delta_0 = (b_n) \in \Lambda_\infty(\alpha)$ *and* $b_0 \neq 0$.

Example 12. *The conditions of the proposition 9 are fulfilled, for instance, if* $\lambda_n = 1$ *or* $\lambda_n = \frac{1}{n!}$ *and* $a_n^k = e^{n^\alpha k}$, $\alpha > 0$.

Two continuous operators commuting with J_λ commute with each other [2] but the converse is not true. For example, take an operator given by an infinite two-block matrix

$$\begin{pmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{0,1} \end{pmatrix}, \qquad a_{0,1}, a_{1,0} \neq 0, \quad a_{0,0} \neq a_{1,1}$$

and the operator J_λ^2. We show that for certain spaces the result is true.

Theorem 13. *Let* T *be a linear operator from* $\lambda^p(A)$ *to* $\lambda^p(A)$, $p = 0$, $p \in [1, +\infty)$ *commuting with* J_λ, $T = \sum_{n=0}^{\infty} \frac{b_n}{\lambda_n} J_\lambda^n$ *and* $\left\{ \lambda_n U^n \left(\frac{b_{j+1}}{\lambda_{j+1}} \right)_{j=0}^{\infty} \right\}_{n=0}^{\infty}$ *is a basis of* $\lambda^1(A)$. *Then any continuous linear operator* S *on* $\lambda^p(A)$, *commuting with* T, *commutes with* J_λ.

Proof. It is similar to the proof of theorem 3.5 in [20].

4 Gončarov polynomials in a nuclear Köthe space

Conditions for the generalized Gončarov polynomials to be a basis in the nuclear space spaces $\lambda^1(A)$ are given.

Given a sequence of complex numbers $(z_n)_{n=0}^{\infty}$, the Gončarov polynomials $G_n(z; z_0, \ldots, z_{n-1})$ are recursively defined by

$$\begin{aligned} G_0(z) &= 1 \\ G_1(z; z_0) &= z - z_0 \\ &\cdots \\ G_n(z; z_0, \ldots, z_{n-1}) &= \frac{z^n}{n!} - \sum_{k=0}^{n-1} \frac{z_k^{n-k}}{(n-k)!} G_k(z; z_1, \ldots, z_{k-1}). \end{aligned}$$

The generalized Gončarov polynomials $Q_n(z; z_0, \ldots, z_{n-1})$ are given by

$$\begin{aligned} Q_0(z) &= 1 \\ G_1(z; z_0) &= d_1(z - z_0) \\ &\cdots \\ Q_n(z; z_0, \ldots, z_{n-1}) &= d_n z^n - \sum_{k=0}^{n-1} d_{n-k} z_k^{n-k} Q_k(z; z_1, \ldots, z_{k-1}) \end{aligned}$$

where (d_n) is a sequence of positive real numbers.

Recall that if X is a locally convex space, a biorthogonal system $\{e_i, f_i\}$, $e_i \in X$, $f_i \in X'$, $f_i(e_j) = \delta_{ij}$, is complete, if the finite linear combinations of (e_i) are dense in X, see [14].

If we define the functionals D_m, L_m, $m \geq 0$ on $\mathscr{H}(\mathbb{D}_R)$ by

$$\begin{aligned} D_m(f(z)) & \sum_{n=m}^{\infty} x_n \frac{n!}{(n-m)!} z_m^{n-m} \\ L_m(f(z)) &= \sum_{n=m}^{\infty} x_n \frac{d_{n-m}}{d_n} z_m^{n-m}, \end{aligned} \qquad f(z) = \sum_{n=0}^{\infty} x_n z^n \in \mathscr{H}(\mathbb{D}_R),$$

then $\{G_m(z; z_0, z_1, \ldots, z_{m-1}); D_m\}_{m=0}^{\infty}$ and $\{Q_m(z; z_0, z_1, \ldots, z_{m-1}); L_m\}_{m=0}^{\infty}$ are biorthogonal systems for $\mathscr{H}(\mathbb{D}_R)$.

Theorem 14 (c.f. [8]). *If $\lambda^1(A)$ is nuclear, a complete biorthogonal system, (e_i, f_i), $f_i = (f_{i,j})$, is a Schauder basis for $\lambda^1(A)$ if and only if $\forall k \in \mathbb{N}$ there exists $r = r(k) \in \mathbb{N}$ such that:*

$$\sup_{i,j} \left(\frac{|f_{i,j}|}{a_j^r} \|e_i\|_k \right) < \infty.$$

Theorem 15 (c.f. [9]). *Let (t_k) be a sequence such that $t_k < t_{k+1}$ and $\lim_{k\to\infty} t_k = R$, $0 < R \leq \infty$. The Gončarov polynomials $G_n(z; z_0, \ldots, z_{n-1})$ are a Schauder basis in $\mathscr{H}(\mathbb{D}_R)$, if and only if $\forall k \in \mathbb{N}$, there exists $r = r(k)$ such that*

$$\sup_{n\geq 0} \sup_{m\geq n} \left\{ \frac{m!|z_n|^{m-n}}{(m-n)!(t_r)^m} \sum_{j=0}^{n} \frac{(t_k)^j}{j!} \left| G_{n-j}(0; z_j, \ldots, z_{n-1}) \right| \right\} < \infty.$$

Theorem 16 (c.f. [10]). *The generalized Gončarov polynomials $Q_n(z; z_0, \ldots, z_{n-1})$ are a basis in $\mathscr{H}(\mathbb{D}_R)$, $0 < R \leq \infty$, if and only if $\forall k \in \mathbb{N}$, $\exists r = r(k)$ such that*

$$\sup_{n\geq 0} \sup_{m\geq n} \left\{ \frac{d_{m-n}}{d_m (t_r)^m} |z_n|^{m-n} \sum_{j=0}^{n} d_j (t_k)^j \left| Q_{n-j}(0; z_j, \ldots, z_{n-1}) \right| \right\} < \infty.$$

The generalized Gončarov polynomials $\{Q_n(z; z_0, \ldots, z_{n-1})\}_{n=0}^{\infty}$ are a complete system in a nuclear space $\lambda^1(A)$ and $L_n \in (\lambda^1(A))'$ if and only if

$$\sup_{m\geq n} \left(\frac{d_{m-n}}{d_m a_m^r} |z_n|^{m-n} \right) < \infty.$$

Proposition 17. *If $\lambda^1(A)$ is a nuclear space, the generalized Gončarov polynomials $\{Q_n(z; z_0, \ldots, z_{n-1})\}_{n=0}^{\infty}$ are a basis in $\lambda^1(A)$ if and only if $\forall k \in \mathbb{N}$, $\exists r = r(k) \in \mathbb{N}$ such that:*

$$\sup_{n\geq 0} \left\{ \sup_{m\geq n} \left(\frac{d_{m-n}}{d_m a_r^m} \left| z_n^{m-n} \right| \right) \sum_{j=0}^{\infty} \left| Q_{n-j}(0; z_j, \ldots, z_{n-1}) d_j a_j^k \right| \right\} < \infty.$$

Proof. Follows easily from Theorem 14.

Acknowledgements

References

[1] J.L Frank and J.K Shaw, (1974), *Abel-Gončarov Polynomial Expansions*, Journal of Approximation Theory 10, No 1, 6-22. doi 10.1016/0021-9045(74)90092-6

[2] R. Gellar, (1969), *Operators Commuting with a Weighted Shift*, Proceedings of the American Mathematical Society, 23, No 3, 538-545. doi 10.2307/2036580

[3] R. Gellar, (1969), *Cyclic Vectors and Parts of the Spectrum of a Weighted Shift*, Transactions of the American Mathematical Society, 146, 69-85. doi 10.2307/1995160

[4] R. Gellar, (1974/1975), *A Generalization of Weighted Shifts*, Indiana University Mathematics Journal, 24, 259-264.

[5] S. Grabiner, (1975), *Weighted Shifts and Banach Algebras of Power Series*, American Journal of Mathematics, 97, No 1, 16-42. doi 10.2307/2373659

[6] S. Grabiner, (1986), *Dense subspaces of entire functions*, The Michigan Mathematical Journal, 33, 417-422.

[7] S. Grabiner, (1988), *Convergent expansions and bounded operators in the umbral calculus*, Advances in Mathematics, 72, No 1, 132-167. doi 10.1016/0001-8708(88)90020-5

[8] F. Haslinger, (1978), *Complete biorthogonal systems in nuclear (F)-spaces*, Mathematische Nachrichten 83, 305-310. doi 10.1002/mana.19780830125

[9] F. Haslinger, (1980), *Abel-Gončarov Polynomial Expansions in Spaces of Holomorphic Functions*, Journal of the London Mathematical Society s2-21, No 3, 487-495. doi 10.1112/jlms/s2-21.3.487

[10] F. Haslinger, (1979), *Generalized Abel-Gončarov bases in spaces of holomorphic functions*, Journal of Approximation Theory, 27, No 4, 297-308. doi 10.1016/0021-9045(79)90119-9
[11] I. I. Ibragimov and N. I. Nagnibida, (1975), *The Matrix Method and Quasi-Power Bases in the Space of Analytic Functions in a Disc*, Russian Mathematical Surveys, 30, No 6, 107-154. doi 10.1070/RM1975v030n06ABEH001533
[12] R. Lorentz, S. Tringali and C. H. Yan, *Generalized Goncarov polynomials*, to appear in S. Butler, J. Cooper, and G. Hurlbert (eds.), Connections in Discrete Mathematics: A Celebration of the Work of Ron Graham, Cambridge University Press. arXiv 1511.04039
[13] M. Maldonado and J. Prada, (2006), *Weighted shift operators on Köthe spaces*, Mathematische Nachrichten 279, No 1-2, 188-197. doi 10.1002/mana.200310355
[14] R. Meise and D. Vogt, (1997), *Introduction to Functional Analysis*, The Clarendon Press, Oxford University Press, Oxford.
[15] N. K. Nikol'skiĭ, (1967), *Invariant Subspaces of Weighted Shift Operators*, Mathematics of the USSR-Sbornik, 3, No 2, 159-176. doi 10.1070/SM1967v003n02ABEH002367
[16] N. K. Nikol'skiĭ, (1967), *The unicellularity and nonunicellularity of weighted shift operators*, Soviet Mathematics Doklady, 8, 91-94.
[17] N. K. Nikol'skiĭ, (1968), *Basicity and unicellularity of weighted shift operators*, Mathematics of the USSR-Izvestiya, 2, 1077-1089. doi 10.1070/IM1968v002n05ABEH000703
[18] N. K. Nikol'skiĭ, (1970), *Spectral Synthesis for a Shift Operator and Zeros in certain Classes of Analytic Functions Smooth Up to the Boundary*, Soviet Mathematics Doklady, 11, 206-209.
[19] J. Prada, (2002), *Delta Operators on Sequence Spaces*, Scientiae Mathematicae Japonicae, 55, No 2, 223-231.
[20] J. Prada, (2004), *Operators commuting with the shift on sequence spaces*, International Journal of Mathematics and Mathematical Sciences, 43, Volume 2004, 2273-2278. doi 10.1155/S0161171204310124