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Poisson and symplectic reductions of 4–DOF isotropic oscillators. The van der Waals system as benchmark

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Abstract

This paper is devoted to studying Hamiltonian oscillators in 1:1:1:1 resonance with symmetries, which include several models of perturbed Keplerian systems. Normal forms are computed in Poisson and symplectic formalisms, by mean of invariants and Lie-transforms respectively. The first procedure relies on the quadratic invariants associated to the symmetries, and is carried out using Gröbner bases. In the symplectic approach, hinging on the maximally superintegrable character of the isotropic oscillator, the normal form is computed *a la* Delaunay, using a generalization of those variables for 4-DOF systems. Due to the symmetries of the system, isolated as well as circles of stationary points and invariant tori should be expected. These solutions manifest themselves rather differently in both approaches, due to the constraints among the invariants versus the singularities associated to the Delaunay chart.

Taking the generalized van der Waals family as a benchmark, the explicit expression of the Delaunay normalized Hamiltonian up to the second order is presented, showing that it may be extended to higher orders in a straightforward way. The search for the relative equilibria is used for comparison of their main features of both treatments. The pros and cons are given in detail for some values of the parameter and the integrals.

Keywords: Hamiltonian systems; isotropic oscillator; normal form; singular reduction; relative equilibria.

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1 Introduction

The use of computer algebra systems for normal forms computations is considered at present a routine operation. As a general reference see e.g. Sanders *et al.* [36] and Meyer *et al.* [32]. Nevertheless when we deal with special classes of differential equations, like Poisson or Hamiltonians systems which is our case, it

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is advisable to employ specific transformations as well as tailored variables for those problems [32], mostly connected with the symmetries that those systems might possess. More precisely we are interested in perturbed isotropic oscillators in four dimensions (or perturbed harmonic oscillators in 1:1:1:1 resonance). For those Hamiltonian systems, Sanders *et al.* [36] explain that the 1:…:1 resonance is one of the more complicated, due to large number of terms in the normal form. This proves to be a key issue in computations of higher order approximations, which is needed in bifurcation analysis for some values of the parameters. For this reason, any strategy developed to reduce the algebra involved in the normal form process, as well as in the subsequent analysis built on it, like relative equilibria and their bifurcations, is something really desirable. As we will see, we have to confront with systems of polynomial equations with parameters, which represent a real challenge, even with computer algebra assistance.

Continuing previous work [9, 25, 26] in the 1:1:1 resonance, and also [17, 20, 21] in relation to 1:1:1:1 resonance, we consider in $\mathbb{R}^4 \times \mathbb{R}^4$, the symplectic form $w = d\mathbf{Q} \wedge d\mathbf{q}$ and a Hamiltonian function

$$\mathcal{H} = \mathcal{H}_2 + \varepsilon \mathcal{P}(q, Q; \beta) \tag{1}$$

where

$$\mathcal{H}_2 = \frac{1}{2}(Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2) + \frac{1}{2}\omega^2(q_1^2 + q_2^2 + q_3^2 + q_4^2) \tag{2}$$

defines the isotropic oscillator, with ω a positive constant and ε is an small parameter. In the first part of the paper we simplify expressions assuming $\omega = 1$. The function \mathcal{P} is called the perturbation, where β is a parameter vector, which may include also ε . Moreover, following the same notation as in [21], we consider that systems defined by Hamiltonian function Eq. (1) have two first integrals in involution given by

$$\begin{aligned} \Xi &= q_1 Q_2 - Q_1 q_2 + q_3 Q_4 - Q_3 q_4, \\ L_1 &= q_3 Q_4 - Q_3 q_4 - q_1 Q_2 + Q_1 q_2, \end{aligned} \tag{3}$$

associated to which we have rotational symmetries. Notice that these integrals are precisely the constrains that should be imposed in order to establish the connection between the isotropic and Kepler problem, see [30]. Moreover, this connection will allow us to make use of appropriate coordinates in Section 3 in order to perform a normalisation.

Although some of the methods and techniques considered in the paper may be applied to a large family of Hamiltonian systems defined by (1), to give details of those processes we focus on the uniparametric family of Hamiltonian systems defined by

$$\mathcal{H}_\beta(Q, q) = \mathcal{H}_2 + \varepsilon \mathcal{H}_6, \tag{4}$$

where

$$\begin{aligned} \mathcal{H}_6(Q, q; \beta) &= (q_1^2 + q_2^2 + q_3^2 + q_4^2) \\ &\times \left(\beta^2 (q_1^2 + q_2^2 - q_3^2 - q_4^2)^2 + 4 (q_1^2 + q_2^2) (q_3^2 + q_4^2) \right) \end{aligned} \tag{5}$$

where β is now a real parameter. For $\beta = 1$ we have a central potential, *i.e.* an integrable system. When the system restricts to the manifold $\Xi = 0$ then it is equivalent to the model for the hydrogen atom subject to a generalized van der Waals potential. For $\beta = 0$ this system reduces to the model for the quadratic Zeeman effect. When $\beta = \sqrt{2}$ we have the van der Waals system. For this reason we have named the system as the *generalized van der Waals 4-D oscillator*. A search (see [17]) for some special solutions of the Hamiltonian system defined by the function (4) reveals that there are invariant 2-tori associated to the rotational integrals (3), which include straight-line orbits over the configuration space.

Normalisation and reduction using invariants and Lie-transforms are approached in both Poisson and symplectic formalisms. The Hamiltonian (4) defines a four-degree-of-freedom system with corresponding phase space $T^*\mathbb{R}^4$. It is endowed with two symmetries given by \mathcal{H}_2 and Ξ . Therefore, we have that, without any

normalisation process, a 2-DOF reduction is performed on the system by means of the invariants related to the symmetries \mathcal{H}_2 and Ξ . This procedure (see [20], [8]) is based on previous work of the authors. Enlarging previous studies on the 1:1:1 resonance [9, 26], the Hamiltonian (4) is put into normal form with respect to \mathcal{H}_2 and Ξ . After the first reduction of the phase space by the \mathcal{H}_2 symmetry we obtain a reduced phase space given by $\mathbb{C}\mathbb{P}^3$ and the second reduction with respect to Ξ leaves us with a twice reduced space isomorphic to $S^2_{n+\xi} \times S^2_{n-\xi}$, where $H_2 = n$, $\Xi = \xi$. However, further reduction is accomplished by considering the truncated normal form with respect to L_1 . Thus, a system is obtained that is invariant under the \mathbb{S}^1 -actions corresponding to \mathcal{H}_2 , Ξ , and L_1 . These three actions together generate a \mathbb{T}^3 -action and the whole reduction process leads to a one-degree-of-freedom system.

Hinging on the maximally superintegrable character of the isotropic oscillator, the symplectic reduction is carried out *a la* Delaunay using a generalization of those variables to 4-DOF recently proposed in [24]. Based on the normalized equations, our studies on this system focus on relative equilibria and their evolution with the parameter β . Although first order normalization (averaging) is quite generic, the fact that the family includes some integrable cases, makes it necessary to perform higher order normalization for the stability analysis of those cases.

The paper is organized as follows. Section 2 summarizes part of the work done in [18], the three steps of the Poisson reduction are presented with the invariant functions involved, as well as the corresponding reduced Hamiltonian of our model at each step. In Section 3 we proceed likewise carrying out the reduction, although in different order, by three symplectic transformations: Projective Euler and Andoyer as well as 4-D Delaunay. They give again, but now in the symplectic frame, the 1-DOF thrice reduced Hamiltonian system in the open domain where those charts are defined. Once carried out the total reduction by the symplectic variables in Section 3, Section 4 focuses on the computation of the normal form by Lie transform which, taking advantage of the periodic character of the unperturbed flow, which allows it to solve the homological equation just by quadratures. Finally Section 5 gathers the analysis on relative equilibria which enlarges the results presented in previous papers, in particular the search for periodic orbits.

2 Poisson approach. Normalization and reduction

There is a large literature which develops both theoretical and computational aspects of normal forms, see for example [2, 31, 32, 36]. In this section we briefly describe several normal forms for first order of system (4) for $\omega = 1$ thoroughly presented in [18]. In such a work, we used the computer algebra system Maple as software, and as main computational tool, Gröbner bases. A finite set G of multivariate polynomials is called a Gröbner basis (with respect to a term order) of the ideal generated by G if the remainder on division by G is unique. In our computations the graded reverse lexicographic order (grevlex) was considered because it is usually the most efficient for computations (for more details see [3] and [4]). The reader is assumed to distinguish the truncation order in the Lie Transform method from monomial orders in the multivariate polynomial ring.

The non perturbed model (2) is geometrically reduced in [38] where the connection between the Kepler and the harmonic oscillator systems is revisited. In the present work we focus on (4) and we review the constructive geometric reduction in stages performed in [18]. Only the final steps of such a reduction process are presented along this section. In the Poisson approach the first reduction is done with respect to the H_2 symmetry. This is a regular reduction and the reduced phase space is homeomorphic to $\mathbb{C}\mathbb{P}^3$ (see [8]). Then we carry out a second reduction whose resulting orbit space is stratified by four dimensional leaves isomorphic to $\mathbb{S}^2 \times \mathbb{S}^2$, and two singular strata isomorphic to \mathbb{S}^2 . Finally, after discarding de higher-order terms, the system in normal form has another integral, i.e., L_1 which allows a third reduction leading to a 1-DOF system on the *thrice reduced phase space*. Depending on the relative value of the integrals, the thrice reduced space is isomorphic or homomorphic to a 2-sphere, containing one or two singular points. Besides there are other singular reduced phase spaces consisting of a single point (see Fig. 2 and [21]).

2.1 Normalization with respect to the oscillator symmetry X_{H_2}

Given the action associated to the uniparametric group defined by X_{H_2} , by using canonical complex variables, (see [21] for details) it follows that the algebra of polynomial invariants under that action is generated by

$$\begin{aligned} \pi_i &= Q_i^2 + q_i^2, & i &= 1, 2, 3, 4 \\ \pi_5 &= Q_1 Q_2 + q_1 q_2, & \pi_6 &= Q_1 Q_3 + q_1 q_3, & \pi_7 &= Q_1 Q_4 + q_1 q_4, \\ \pi_8 &= Q_2 Q_3 + q_2 q_3, & \pi_9 &= Q_2 Q_4 + q_2 q_4, & \pi_{10} &= Q_3 Q_4 + q_3 q_4, \\ \pi_{11} &= q_1 Q_2 - Q_1 q_2, & \pi_{12} &= q_1 Q_3 - Q_1 q_3, & \pi_{13} &= q_1 Q_4 - Q_1 q_4, \\ \pi_{14} &= q_2 Q_3 - Q_2 q_3, & \pi_{15} &= q_2 Q_4 - Q_2 q_4, & \pi_{16} &= q_3 Q_4 - Q_3 q_4. \end{aligned} \tag{6}$$

The X_{H_2} normal form up to first order in ε is expressed in those invariants as

$$\overline{\mathcal{H}} = \mathcal{H}_2 + \varepsilon \overline{\mathcal{H}}_6 \tag{7}$$

where $\mathcal{H}_2 = (\pi_1 + \pi_2 + \pi_3 + \pi_4) / 2 = n$ and

$$\begin{aligned} \overline{\mathcal{H}}_6 &= \frac{1}{2} [(1 - 4\beta^2)n(\pi_{15}^2 + \pi_{14}^2 + \pi_{13}^2 + \pi_{12}^2) \\ &\quad + 2(\beta^2 - 1)(\pi_{11}^2(\pi_4 + \pi_3) - \pi_{16}^2(\pi_3 + \pi_4)) + \beta^2 n(5n^2 - 3\pi_{11}^2) \\ &\quad + 5(1 - \beta^2)n(\pi_9^2 + \pi_8^2 + \pi_7^2 + \pi_6^2) + (\beta^2 - 4)n\pi_{16}^2] \end{aligned} \tag{8}$$

The reduction is now performed using the orbit map

$$\rho_\pi : \mathbb{R}^8 \rightarrow \mathbb{R}^{16}; (q, Q) \rightarrow (\pi_1, \dots, \pi_{16}).$$

The image of this map is the orbit space for the H_2 -action, the images of the level surfaces $H_2(q, Q) = n$ under ρ_π are the reduced phase spaces which are isomorphic to $\mathbb{C}\mathbb{P}^3$. Since the Hamiltonian is invariant respect to \mathcal{H}_2 , it can be expressed in invariants and therefore naturally lifts to a function on \mathbb{R}^{16} , which, on the reduced phase spaces, leads to the reduced Hamiltonian.

However, in the following we will not use the invariants π_i ; instead, we rely on (K_i, L_j, J_k) invariants introduced in Egea [20] by the following change of coordinates,

$$\begin{aligned} H_2 &= \frac{1}{2}(\pi_1 + \pi_2 + \pi_3 + \pi_4), & J_1 &= \frac{1}{2}(\pi_1 - \pi_2 - \pi_3 + \pi_4), \\ \Xi &= \pi_{16} + \pi_{11}, & J_2 &= \frac{1}{2}(\pi_1 - \pi_2 + \pi_3 - \pi_4), \\ K_1 &= \frac{1}{2}(-\pi_1 - \pi_2 + \pi_3 + \pi_4), & J_3 &= \pi_8 + \pi_7, \\ K_2 &= \pi_8 - \pi_7, & J_4 &= \pi_5 + \pi_{10}, \\ K_3 &= -\pi_6 - \pi_9, & J_5 &= \pi_5 - \pi_{10}, \\ L_1 &= \pi_{16} - \pi_{11}, & J_6 &= \pi_6 - \pi_9, \\ L_2 &= \pi_{12} + \pi_{15}, & J_7 &= \pi_{12} - \pi_{15}, \\ L_3 &= \pi_{14} - \pi_{13}, & J_8 &= \pi_{14} + \pi_{13}. \end{aligned}$$

The *first integrals* (see Eq. 3) are now among the invariants defining the image.

The first order normal form in these invariants is

$$\begin{aligned} \overline{\mathcal{H}}_\Xi &= \frac{1}{2} \left[n \left(5K_2^2 + 5K_3^2 + 2L_1^2 + L_2^2 + L_3^2 + \beta^2(5K_1^2 + L_2^2 + L_3^2) \right) \right. \\ &\quad \left. - ((4 + \beta^2)(K_2L_2 + K_3L_3) + (2 + 3\beta^2)K_1L_1) \xi \right]. \end{aligned} \tag{9}$$

The reduction of the H_2 action may now be performed through the orbit map

$$\rho_{K,L,J} : \mathbb{R}^8 \rightarrow \mathbb{R}^{16}; (q, Q) \rightarrow (H_2, \dots, J_8).$$

Note that on the orbit space we have the reduced symmetries due to the reduced actions given by the reduced flows of X_Ξ and X_{L_1} .

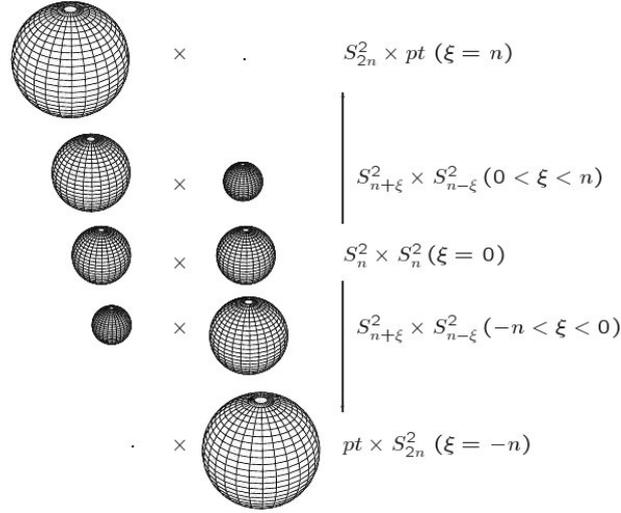


Fig. 1 Double reduced spaces $S_{n+\xi}^2 \times S_{n-\xi}^2$ for different values of the integral ξ

2.2 Toroidal reduction over $\mathbb{C}\mathbb{P}^3$ with respect to the rotational symmetry Ξ

Let ρ be the S^1 -action generated by the Poisson flow of Ξ over $\mathbb{C}\mathbb{P}^3$. The functions

$$H_2, \quad \Xi, \quad L_1, \quad L_2, \quad L_3, \quad K_1, \quad K_2, \quad K_3,$$

are ρ -invariants over $\mathbb{C}\mathbb{P}^3$. This, in turn, leads us to the orbit mapping

$$\rho_2 : \mathbb{R}^{16} \rightarrow \mathbb{R}^8; \quad (\pi_1, \dots, \pi_{16}) \rightarrow (K_1, K_2, K_3, L_1, L_2, L_3, H_2, \Xi).$$

The orbit space $\rho_2(\mathbb{C}\mathbb{P}^3)$ is defined as a six dimensional algebraic variety in \mathbb{R}^8 by the following relations

$$K_1^2 + K_2^2 + K_3^2 + L_1^2 + L_2^2 + L_3^2 = H_2^2 + \Xi^2, \quad K_1 L_1 + K_2 L_2 + K_3 L_3 = H_2 \Xi \quad (10)$$

The reduced phase spaces are obtained by setting $H_2 = n$, $\Xi = \xi$ and then the second reduced space is isomorphic to $S_{n+\xi}^2 \times S_{n-\xi}^2$ (see Fig. 1). When $\xi = 0$, it corresponds to the first reduced space of Keplerian systems by the energy. The second reduced Hamiltonian up to first order, modulo a constant takes the form

$$\begin{aligned} \overline{\mathcal{H}}_\Xi = & \frac{1}{2} [n (5K_2^2 + 5K_3^2 + 2L_1^2 + L_2^2 + L_3^2 + \beta^2 (5K_1^2 + L_2^2 + L_3^2)) \\ & - ((4 + \beta^2)(K_2 L_2 + K_3 L_3) + (2 + 3\beta^2)K_1 L_1) \xi] \end{aligned}$$

2.3 Reduction by $L_1 = l$. Thrice reduced space $V_n \zeta l$

To further reduce from $S_{n+\xi}^2 \times S_{n-\xi}^2$ to $V_n \zeta l$, one divides out the S^1 -action, ρ_2 , generated by the Poisson flow defined by $L_1 = \pi_{16} - \pi_{11}$ over $S_{n+\xi}^2 \times S_{n-\xi}^2$. The 8 invariants for the L_1 action on \mathbb{R}^8 are

$$\begin{aligned} M &= \frac{1}{2}(K_2^2 + K_3^2 + L_2^2 + L_3^2), & N &= \frac{1}{2}(K_2^2 + K_3^2 - L_2^2 - L_3^2), \\ Z &= K_2 L_2 + K_3 L_3, & S &= K_2 L_3 - K_3 L_2, \\ K &= K_1, & \mathcal{H}_2, & \quad \Xi, & L_1. \end{aligned} \quad (11)$$

There are 2 + 3 relations defining the third reduced phase space

$$\begin{aligned} K^2 + L_1^2 + 2M &= n^2 + \xi^2, & M^2 - N^2 &= Z^2 + S^2, \\ KL_1 + Z &= n\xi, & \mathcal{H}_2 &= n, & \Xi &= \xi, & L_1 &= l. \end{aligned} \quad (12)$$

{,}	M	N	Z	S	K	L ₁
M	0	4KS	0	-4KN	0	0
N	-4KS	0	-4L ₁ S	-4(KM - L ₁ Z)	4S	0
Z	0	4L ₁ S	0	-4L ₁ N	0	0
S	4KN	4(KM - L ₁ Z)	4L ₁ N	0	-4N	0
K	0	-4S	0	4N	0	0
L ₁	0	0	0	0	0	0

Table 1 Poisson structure in (M,N,Z,S,K,L₁) invariants

The Poisson structure of the variables S, K and N is given in Table 1. For more details see [18]. Then, we have defined the orbit map

$$\rho_2 : \mathbb{R}^6 \rightarrow \mathbb{R}^6; (K_1, K_2, K_3, L_1, L_2, L_3) \rightarrow (M, N, Z, S, K, L_1).$$

When we fix a value of L₁ = l, relations (12) define the *thrice reduced space*

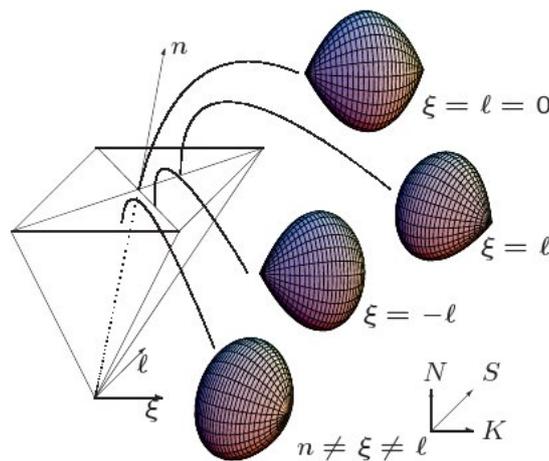


Fig. 2 Thrice reduced space over the space of integrals. The vector (K,N,S) represents the coordinates. The axis of symmetry of the reduced space is the K direction.

$$V_{n\xi l} = \{ (K, S, N) \mid 4N^2 + 4S^2 = f(K), \\ f(K) = ((n + \xi)^2 - (K + l)^2)((n - \xi)^2 - (K - l)^2) \}$$

which is a surface of revolution, obtained by rotating $\sqrt{f(K)}$ around the axis K, . Thus the shape of the reduced phase space, given in Fig. 2, is determined by the positive part of f(K). Since

$$f(K) = (K + n + \xi + l)(K - n - \xi + l)(K - n + \xi - l)(K + n - \xi - l),$$

the roots are

$$k_1 = -l - n - \xi, \quad k_2 = l + n - \xi, \quad k_3 = l - n + \xi, \quad k_4 = -l + n + \xi .$$

So $f(K)$ is positive (or zero) in the subsequent intervals of K :

$$\begin{aligned}
 l < \xi, -l < \xi & \quad k_1 < k_3 < k_2 < k_4 & \quad K \in [k_3, k_2] \\
 l > \xi, -l < \xi & \quad k_1 < k_3 < k_4 < k_2 & \quad K \in [k_3, k_4] \\
 l < \xi, -l > \xi & \quad k_3 < k_1 < k_2 < k_4 & \quad K \in [k_1, k_2] \\
 l > \xi, -l > \xi & \quad k_3 < k_1 < k_4 < k_2 & \quad K \in [k_1, k_4]
 \end{aligned} \tag{13}$$

The Hamiltonian on the third reduced phase space is

$$\begin{aligned}
 \overline{\mathcal{H}}_{\Xi, L_1} &= \frac{3n}{4} (3\beta^2 - 2) K^2 + \xi l (1 - \beta^2) K \\
 &+ \frac{n}{2} (4 - \beta^2) N + n^3 \left(\frac{3}{2} + \frac{\beta^2}{4} \right) - (l^2 + \xi^2) \left(\frac{\beta^2}{2} + 1 \right) \frac{n}{2}
 \end{aligned}$$

Note that the reduced phase spaces as well as the Hamiltonian are invariant under the discrete symmetry $S \rightarrow -S$. To see how to exploit it to obtain a full reduction see [10], [19] and [29]. We choose not to further reduce our reduced phase space with respect to these discrete symmetries, because the three dimensional picture makes it easy to access information about the reduced orbits, and in this way one does not introduce additional critical points (fixed points) which need special attention.

In (K, N, S) -space the energy surfaces are parabolic cylinders. Notice that for $\beta^2 = 2/3$, the function \mathcal{H} is linear in the variable space (K, N, S) . Likewise for $\beta^2 = 1$, \mathcal{H} modulo constants is independent of ξ and l . Moreover for $\beta^2 = 4$, \mathcal{H} is only a function of K . Since $(V_{n\xi l}, \{\cdot, \cdot\}_3, \overline{\mathcal{H}}_{\Xi, L_1})$ is a Lie-Poisson system, the corresponding dynamics is given by

$$\begin{aligned}
 \frac{dK}{dt} &= 2n(\beta^2 - 4)S, \\
 \frac{dN}{dt} &= 2[3n(3\beta^2 - 2)K + 2\xi l(1 - \beta^2)]S, \\
 \frac{dS}{dt} &= n(\beta^2 - 4)(K^2 - (\xi^2 + l^2 + n^2))K \\
 &\quad - (3\beta^2 - 2)[6nKN + 4\xi l(\beta^2 - 1)N + 2ln^2\xi].
 \end{aligned} \tag{14}$$

This system can be integrated by means of elliptic functions, but after a classification of the different types of flows is made, as functions of the integrals and the parameter. Only then we will be ready for the integration of a specific initial value problem. Note that in the search of relative equilibria at the thrice reduced level, the geometry involved is really helpful; the intersection of the Hamiltonian surfaces with the reduced phase space gives the trajectories of the reduced system. Then, tangency of the Hamiltonian surface with the reduced phase spaces provides relative equilibria that generically correspond to three dimensional tori in the original phase space. Details on the analysis of this system, focusing on relative equilibria related to $\beta^2 = 0$, $\xi = l$ and the case of physical interest $\xi = 0$, are contained in Díaz *et al.* [18]. The reader should take into account that β^2 is renamed in [18] as λ .

3 Symplectic charts for 4-D isotropic oscillators

In our study of perturbed oscillators, we switch now from Poisson to symplectic formulation. More precisely, our goal is to obtain the normal form of isotropic oscillators by Lie transforms, in the presence of symmetries.

Different charts are used for the symplectic treatment of the harmonics oscillators. Among them we find complex notation [8], action-angle variables of Poincaré type [36], [25], or variations of them in the case of

resonances [15]. In our case, as we also need to reduce by the actions associated to the rotational symmetries, we have made use of a symplectic chart which incorporates the functions (3) as momenta. It is well known in astrodynamics that the Euler and Andoyer variables carry out this task for classical problem as Kepler and the rigid body. In previous works [5, 27], we extend those variables to the four dimensional case. Moreover the connection between the isotropic and Kepler problems, two maximally superintegrable systems, allows to introduce the Delaunay transformation [14]. The result is the *4-D Delaunay transformation* proposed by one of the authors [24], that we will use in what follows. There is an alternative procedure considering the Lissajous transformation [16], which will be presented elsewhere.

3.1 The family of Euler projective transformations

Let $F(\rho)$ be a smooth real function which is positive in its domain. We consider the family of transformations: $\mathcal{P}\mathcal{E}_F : (\rho, \phi, \theta, \psi) \rightarrow (q_1, q_2, q_3, q_4)$, dubbed as *Projective Euler* variables, given by

$$\begin{aligned} q_1 &= F(\rho) \sin \frac{\theta}{2} \cos \frac{\phi - \psi}{2}, & q_3 &= F(\rho) \cos \frac{\theta}{2} \sin \frac{\phi + \psi}{2}, \\ q_2 &= F(\rho) \sin \frac{\theta}{2} \sin \frac{\phi - \psi}{2}, & q_4 &= F(\rho) \cos \frac{\theta}{2} \cos \frac{\phi + \psi}{2}, \end{aligned} \quad (15)$$

with $(\rho, \phi, \theta, \psi) \in \mathbb{R}^+ \times [0, 2\pi) \times (0, \pi) \times (-\frac{\pi}{2}, \frac{\pi}{2})$. For $F(\rho) = 1$, this transformation defines *Euler parameters* as functions of Euler angles. In this paper, we only consider the case $F(\rho) = \sqrt{\rho}$.

The canonical extension associated to the transformation (15) is readily obtained as a Mathieu transformation, which satisfies $\sum Q_i dq_i = P d\rho + \Theta d\theta + \Psi d\psi + \Phi d\phi$. The relations among the momenta are given by

$$\begin{aligned} P &= \frac{1}{2\sum q_i^2} (q_1 Q_1 + q_2 Q_2 + q_3 Q_3 + q_4 Q_4), \\ \Theta &= \frac{(q_1 Q_1 + q_2 Q_2)(q_3^2 + q_4^2) - (q_3 Q_3 + q_4 Q_4)(q_1^2 + q_2^2)}{2\sqrt{(q_1^2 + q_2^2)(q_3^2 + q_4^2)}}, \\ \Psi &= \frac{1}{2} (-q_2 Q_1 + q_1 Q_2 + q_4 Q_3 - q_3 Q_4), \\ \Phi &= \frac{1}{2} (q_2 Q_1 - q_1 Q_2 + q_4 Q_3 - q_3 Q_4), \end{aligned} \quad (16)$$

Later on we will need the inverse transformation given by

$$\begin{aligned} \rho &= q_1^2 + q_2^2 + q_3^2 + q_4^2, \\ \sin \theta &= \frac{2\sqrt{(q_1^2 + q_2^2)(q_3^2 + q_4^2)}}{q_1^2 + q_2^2 + q_3^2 + q_4^2}, & \cos \theta &= \frac{q_3^2 + q_4^2 - q_1^2 - q_2^2}{q_1^2 + q_2^2 + q_3^2 + q_4^2}, \\ \sin \psi &= \frac{q_1 q_3 + q_2 q_4}{\sqrt{(q_1^2 + q_2^2)(q_3^2 + q_4^2)}}, & \cos \psi &= \frac{q_1 q_4 - q_2 q_3}{\sqrt{(q_1^2 + q_2^2)(q_3^2 + q_4^2)}}, \\ \sin \phi &= \frac{q_1 q_3 - q_2 q_4}{\sqrt{(q_1^2 + q_2^2)(q_3^2 + q_4^2)}}, & \cos \phi &= \frac{q_1 q_4 + q_2 q_3}{\sqrt{(q_1^2 + q_2^2)(q_3^2 + q_4^2)}}. \end{aligned} \quad (17)$$

Proposition 1. *The isotropic oscillator \mathcal{H}_2 is expressed by mean of the Euler projective variables as*

$$\mathcal{H}_\omega(\rho, \theta, -, -, P, \Psi, \Theta, \Phi) = \frac{\omega\rho}{2} + 2\rho P^2 + \frac{2}{\rho} \left(\Theta^2 + \frac{\Psi^2 + \Phi^2 - 2\Phi\Psi \cos \theta}{\sin^2 \theta} \right). \quad (18)$$

Proof. Considering any given initial condition for the Hamiltonian $\mathcal{H}_\omega = h$, by applying relations (15) and (16), we obtain, by straight forward computations, the above expression for the isotropic oscillator \mathcal{H}_2 .

Note that the Hamiltonian has two cyclic variables, which manifests the two symmetries associated with our system we have refer in (3)

3.2 From Euler to Projective Andoyer variables

Andoyer [1, 11] symplectic variables are well known in rotational dynamics and recently in attitude and control. Most often they are denoted by $(\lambda, \mu, \nu, \Lambda, M, N)$ or (h, g, ℓ, H, G, L) . In the following, in order to avoid confusion with invariants notation of previous section, we propose to use $(u_1, u_2, u_3, U_1, U_2, U_3)$ for referring to them. Moreover, as in 3-D, it is convenient to introduce the following functions

$$c_1 = \cos \sigma_1 = U_1/U_2, \quad c_2 = \cos \sigma_2 = U_3/U_2, \quad (19)$$

with $s_i = \sqrt{1 - c_i^2}$. Hence the transformation from Euler to Andoyer $(\phi, \theta, \psi, \Phi, \Theta, \Psi) \rightarrow (u_1, u_2, u_3, U_1, U_2, U_3)$ is given by

$$\begin{aligned} \cos \theta &= c_1 c_2 + s_1 s_2 \cos u_2, & \Phi &= U_3, \\ \sin(\psi - u_1) &= \frac{\sin u_2}{\sin \theta} s_2, & \Psi &= U_1, \\ \sin(\phi - u_3) &= \frac{\sin u_2}{\sin \theta} s_1, & \Theta &= U_2 \sqrt{1 - \frac{c_1^2 + c_2^2 - 2c_1 c_2 \cos \theta}{\sin^2 \theta}}. \end{aligned} \quad (20)$$

with $|U_1| < U_2$ and $|U_3| < U_2$. By adding the variables (ρ, P) we get a 4-D set $(\rho, u_1, u_2, u_3, P, U_1, U_2, U_3)$ of symplectic variables that we call ‘Projective Andoyer variables’.

Theorem 2. *In Projective Andoyer variables, the system defined by (2), regularized by $ds = 1/(4\rho) d\tau$, includes the Keplerian system for any value of the integral U_3 .*

Proof. Following Poincaré technique, we first regularize (2) and our new Hamiltonian will be $\mathcal{K} = (\mathcal{H}_\omega - h)/(4\rho)$. Considering now Proposition 1, after some straightforward calculations, the Hamiltonian of the 4-D isotropic oscillator is given by

$$\tilde{\mathcal{K}} = \frac{1}{2} \left(P^2 + \frac{U_2^2}{\rho^2} \right) - \frac{\gamma}{\rho} \quad (21)$$

in the manifold $\tilde{\mathcal{K}} = -\omega/8$ and $\gamma = \frac{h}{4}$. But the above Hamiltonian corresponds to the Kepler system in polar nodal variables, see [13, 28].

3.3 Delaunay symplectic chart

Following [14] we plan to normalize perturbed isotropic systems by Lie-transforms *a la* Delaunay in the following Section. Thus, we still need to implement (in part of the phase space) another canonical transformation of Hamilton-Jacobi type: $\mathcal{D}_\gamma : (L, G, \ell, g) \rightarrow (\rho, u_2, P, U_2)$. The process hinges on the H-J equation built on the function (21). For details of this transformation \mathcal{D}_γ we refer to [14]; here we only include the final expressions. Among the relations defining Delaunay transformation, which can only be given in implicit form, we take

$$\begin{aligned} u_2 &= g + f, & U_2 &= G, \\ \rho &= a(1 - e \cos E), & P &= \frac{L e \sin E}{a(1 - e \sin E)}, \end{aligned} \quad (22)$$

where $a = a(L)$, $e = e(L, G)$ are given by

$$a = L^2/\gamma, \quad \eta = G/L, \quad e = \sqrt{1 - \eta^2}, \quad (23)$$

and f and E are auxiliary angles. The symplectic variable ℓ is related to them by

$$\ell = E - e \sin E, \quad \tan f/2 = \sqrt{(1+e)/(1-e)} \tan E/2. \tag{24}$$

Later on we also need $1/\rho = (1 - e \cos f)/p$, where $p = a\eta^2$, and

$$\rho \cos f = a(\cos E - e), \quad \rho \sin f = a\eta \sin E, \quad d\ell = (1 - e \cos E) dE. \tag{25}$$

Completing the functions of the momenta given above (23), it is convenient to introduce two more state functions

$$w = U_1/L, \quad z = U_3/L. \tag{26}$$

The previous results may be summarized in the following theorem:

Theorem 3. *By composing the three symplectic transformations*

$$\begin{array}{ccc} \begin{pmatrix} q_1, q_2, q_3, q_4 \\ Q_1, Q_2, Q_3, Q_4 \end{pmatrix} & \xrightarrow{\text{Projective Euler}} & \begin{pmatrix} \rho, \phi, \theta, \psi \\ P, \Phi, \Theta, \Psi \end{pmatrix} \\ \downarrow & & \downarrow \text{PA} \\ \begin{pmatrix} \ell, u_1, g, u_3 \\ L, U_1, G, U_3 \end{pmatrix} & \xleftarrow{\text{DeLaunay}} & \begin{pmatrix} \rho, u_1, u_2, u_3 \\ P, U_1, U_2, U_3 \end{pmatrix} \end{array}$$

together with the regularization $ds = 1/(4\rho) d\tau$, the Hamiltonian of the 4-D isotropic oscillator is given by

$$\mathcal{H}_0 = -\frac{\gamma^2}{2L^2} \tag{27}$$

in the manifold $\mathcal{H}_0 = -\omega/8$.

The set of variables $(\ell, g, u_1, u_3, L, G, U_1, U_3)$ is what we call 4-D Delaunay chart. As in 3-D with the classical Delaunay variables, they represent a generalized set of action-angle variables (see [34]). In the next section we show the interest of this symplectic transformation. For the benefit of the reader let us mention that Moser and Zehnder [33] introduced action-angle variables for the Kepler problem in \mathbb{R}^n and called them Delaunay variables. When we restrict to $n = 4$, those variables do not coincide with the set built in this paper.

Finally, in order to compare results of our analysis of relative equilibria, we need to establish the connection between the variables K (Poisson approach) and G (symplectic approach) defining the thrice reduced space. After some algebraic manipulations, we obtain

$$4G^2 = \frac{1}{2}(n^2 + \xi^2 + l^2) - \frac{1}{2}K^2 - N, \tag{28}$$

which expresses G as function of K and N , invariants defining the thrice reduced space, and the first integrals.

4 Delaunay normalization of perturbed 4-D isotropic oscillators

Our goal in this section is to make normalization by Lie transforms [12], using 4-D Delaunay variables. As we will see they allow to express perturbed isotropic oscillators in a ‘convenient’ form, to implement Lie transforms up to higher order in an efficient way. What we mean by convenient is the following.

It is well known that the isotropic oscillator \mathcal{H}_0 , as the Kepler system, defines a maximally superintegrable Hamiltonian system (see Fassò [23]), whose flow is made of periodic orbits. In that case Cushman [6] proved that any smooth function F over its phase space may be decomposed in a unique way into a sum $F = F^o + F^*$ with the following properties: (i) $\{F^o, \mathcal{H}_0\} = 0$. In other words, F^o belongs to the kernel of the Lie derivative

generated by \mathcal{H}_0 : $\mathcal{L}_0 : F \rightarrow \{F, \mathcal{H}_0\}$; (ii) There exists a smooth function \tilde{F} such that $\{\tilde{F}, \mathcal{H}_0\} = F^*$; in other words F^* belongs to the image of the operator \mathcal{L}_0 . In particular when the Lie transform (see [12], [32])

$$\mathcal{L} : (\ell, g, u_1, u_3, L, G, U_1, U_3) \rightarrow (\ell', g', u'_1, u'_3, L', G', U'_1, U'_3)$$

is carried out *a la* Delaunay [14], the *homological equation*, which relates the new Hamiltonian $\sum(\varepsilon^j/j!)\mathcal{H}'_j$ with the generating function $\sum(\varepsilon^j/j!)\mathcal{W}'_j$, is given by

$$\{\mathcal{W}_j, \mathcal{H}_0\} + \mathcal{H}'_j = \tilde{\mathcal{H}}_j, \quad j \geq 1. \tag{29}$$

Then, considering the splitting coming from Cushman theorem, the equation (29) may be solved choosing

$$\mathcal{H}'_j = \langle \tilde{\mathcal{H}}_j \rangle = \frac{1}{2\pi} \int_0^{2\pi} \tilde{\mathcal{H}}_j d\ell, \quad \mathcal{W}_j = \frac{1}{n} \int (\tilde{\mathcal{H}}_j - \langle \tilde{\mathcal{H}}_j \rangle) d\ell, \tag{30}$$

where, according with (27), $\partial \mathcal{H}_0 / \partial L = \gamma^2 / L^3$. The Hamiltonian (1) of perturbed isotropic oscillators in Delaunay variables is given by

$$\mathcal{H}(\ell, g, u_1, u_3, L, G, U_1, U_3) = -\frac{\gamma^2}{2L^2} + \varepsilon \mathcal{P}(\ell, g, u_1, u_3, L, G, U_1, U_3; \beta).$$

and the normalized Hamiltonian up to order k takes the form

$$\mathcal{H}' = -\frac{\gamma^2}{2L^2} + \sum_{j=1}^k \frac{\varepsilon^j}{j!} \mathcal{H}'_j(-, g', u'_1, u'_3, L', G', U'_1, U'_3; \beta) + \mathcal{O}(\varepsilon^{k+1}).$$

The main feature of the process is that, at each order, solving the *homological equation* (29) only involves quadratures.

4.1 Delaunay normalization and symmetries

Apart from the procedure associated to Delaunay normalization we have just mentioned, when we restrict to systems with the symmetries (3), the use of Delaunay chart shows its full advantage. Indeed, in that case u_1 and u_3 are cyclic, in other words, we have

$$\mathcal{H}(\ell, g, -, -, L, G, U_1, U_3) = -\frac{\gamma^2}{2L^2} + \varepsilon \mathcal{P}(\ell, g, -, -, L, G, U_1, U_3; \beta; \varepsilon). \tag{31}$$

In geometric language, the use of Delaunay variables has carried out the toral reduction associated to the actions defined by the symmetries (3), and the function (31) is the reduced Hamiltonian. The Hamiltonian system given by (31) is a 2-DOF system. Thus, the normalized system will take the form

$$\mathcal{H}' = -\frac{\gamma^2}{2L^2} + \sum_{j=1}^k \frac{\varepsilon^j}{j!} \mathcal{H}'_j(-, g', -, -, L', G', U'_1, U'_3; \beta) + \mathcal{O}(\varepsilon^{k+1}). \tag{32}$$

In our case normalizing up to the order needed, after truncating, we will obtain an integrable 1-DOF Hamiltonian system. From now on we drop primes in the variables in order to simplify the expressions.

4.2 Carrying out the normalization. The van der Waals model as a benchmark

In order to implement these normalizations, since we are dealing with polynomial perturbations, from experience with 3-DOF Kepler systems we introduce the auxiliary variable E by using the following proposition.

Proposition 4. *The functions $F(\rho, \theta) = \rho^m \cos^n \theta$, ($m, n \in \mathbb{N}$ and $m \geq n$), expressed in Delaunay variables, could be written as $F(\rho, \theta) = \sum_{i=0}^m (C_i \cos iE + S_i \sin iE)$, where C_i and S_i are given by expressions of the form $\sum_j (c_{ij} \cos jg + s_{ij} \sin jg)$ which belong to the kernel of the Keplerian Lie derivative, and c_{ij}, s_{ij} are rational functions of the momenta.*

Proof. It is straightforward, based on relations given in (20), (22) and (25).

Applying the above result, the Hamiltonian (4) takes the form

$$\mathcal{H} = -\frac{\gamma^2}{2L^2} + \sum_{j=0}^k \frac{\varepsilon^j}{j!} \sum_i (C_{ij} \cos iE + S_{ij} \sin iE) + \mathcal{O}(\varepsilon^{k+1}), \tag{33}$$

where functions $C_{ij} = C_{ij}(g, L, G, U_1, U_3; \beta)$ and $S_{ij} = S_{ij}(g, L, G, U_1, U_3; \beta)$ belong to the kernel of the Lie transform.

Thus, by considering the differential relation (25) and by replacing in (30), we obtain the part in the kernel as follows

$$\begin{aligned} \langle \mathcal{H}_1 \rangle &= \frac{1}{2\pi} \int_0^{2\pi} \mathcal{H}_1 d\ell \\ &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{i=0} (C_{i1} \cos iE + S_{i1} \sin iE) (1 - e \cos E) dE \\ &= C_{01} + C_{11} \cos g + C_{21} \cos 2g. \end{aligned}$$

with $C_{i1} = C_{i1}(L, G, U_1, U_3; \beta)$ given by

$$\begin{aligned} C_{01} &= \frac{1}{16} a^2 (2 + 3e^2) [2 + (\beta^2 - 1)(2c_1^2 c_2^2 + s_1^2 s_2^2)], \\ C_{11} &= -\frac{1}{4} a^2 (\beta^2 - 1) (4 + e^2) e c_1 c_2 s_1 s_2, \\ C_{21} &= \frac{5}{16} a^2 (\beta^2 - 1) e^2 s_1^2 s_2^2. \end{aligned} \tag{34}$$

Then, we solve the first order homological equation (29). The first order normalized Hamiltonian takes the form

$$\mathcal{H} = -\frac{\gamma^2}{2L^2} + \varepsilon (C_{01} + C_{11} \cos g + C_{21} \cos 2g) + \mathcal{O}(\varepsilon^2)$$

The first order \mathcal{W}_1 of the generating function is obtained computing the second quadrature in (30), then,

taking into account the shorthand $\alpha = \beta^2 - 1$, we get the following expression

$$\begin{aligned} \mathcal{W}_1 = & \frac{\alpha^3 L}{96\gamma} (\alpha(2c_1^2 c_2^2 + s^2 s_2^2) + 2) \\ & \left(c_1 (e^3 \sin(3u) - 9e^2 \sin(2E) + 3e(3\eta^2 + 5) \sin(u)) + \alpha s_1 s_2 \right. \\ & + s_1 s_2 (1 - \eta)^2 (-15e \sin(2g - E) + (9 + 6\eta) \sin(2g - 2E) \\ & \quad \left. - e \sin(2g - 3E)) \right. \\ & - s_1 s_2 (1 + \eta)^2 (15e \sin(2g + E) - (9 - 6\eta) \sin(2g + 2E) \\ & \quad \left. + e \sin(2g + 3E)) \right. \\ & + 4c_1 c_2 (1 - \eta) (e^2 \sin(g - 3E) - 3e(3 + \eta) \sin(g - 2E) \\ & \quad \left. + 3(2\eta^3 + \eta^2 + 5) \sin(g - E)) \right. \\ & \left. - 4c_1 c_2 (1 + \eta) (e^2 \sin(g + 3E) - 3e(3 - \eta) \sin(g + 2E) \right. \\ & \quad \left. + 3(-2\eta^3 + \eta^2 + 5) \sin(g + E)) \right). \end{aligned}$$

Analogously we obtain the second order normalization for (32), which will be used in the study of stability. The second order coefficients are

$$\begin{aligned} C_{02} = & -60 \alpha^2 c_2^2 c_1^2 s_1^2 s_2^2 e^6 \\ & + \left(\frac{\alpha^2}{8} (1139 c_2^4 c_1^4 - 1486(c_2^4 c_1^2 + c_2^2 c_1^4) + 579(c_1^4 + c_2^4) \right. \\ & \quad \left. + 1268 c_2^2 c_1^2 - 518(c_1^2 + 2c_2^2) - 61) + 63(\alpha \Lambda_{3,1} - 1) \right) e^4 \\ & + \left(\alpha^2 (2089 c_2^4 c_1^4 - 1756(c_2^2 c_1^4 + c_2^4 c_1^2) + 399(c_1^4 + c_2^4) \right. \\ & \quad \left. + 2048 c_2^2 c_1^2 - 628(c_1^2 + c_2^2) + 229) - 396(\alpha \Lambda_{3,1} - 1) \right) e^2 \\ & + \left(\alpha^2 (557 c_2^4 c_1^4 - 382(c_2^4 c_1^2 + 382 c_2^2 c_1^4) + 17(c_1^4 + c_2^4) \right. \\ & \quad \left. + 308 c_2^2 c_1^2 - 22(c_2^2 + c_1^2) + 5) - 96(\alpha \Lambda_{3,1} - 1) \right), \end{aligned}$$

$$\begin{aligned} C_{12} = & c_2 c_1 s_1 s_2 \left((-155\alpha^2(\Lambda_{2,1} + 41c_1^2 c_2^2 + 36) + 192)e^5 \right. \\ & \quad \left. - (447\alpha^2(\Lambda_{2,1} - 19c_1^2 c_2^2 - 290) + 708\alpha + 2)e^3 \right. \\ & \quad \left. - (67\alpha^2(\Lambda_{2,1} + c_1^2 c_2^2 + 27) - 1200\alpha - 16)e \right), \end{aligned}$$

$$\begin{aligned} C_{22} = & s_1 s_2 \left(-60 \alpha^2 c_2^2 c_1^2 e^6 \right. \\ & \quad \left. + \left(\frac{-\alpha^2}{2} (71\Lambda_{5,2} + 8c_1^2 c_2^2 - 14) - 57\alpha \right) e^4 \right. \\ & \quad \left. + (\alpha^2 (-147\Lambda_{2,1} + 25c_1^2 c_2^2 - 66) + 486\alpha + 3) e^2 \right), \end{aligned}$$

$$C_{32} = c_1 s_1^3 c_2 s_2^3 \alpha^2 (85e^5 + 10e^3),$$

$$C_{42} = -\frac{95}{8} \alpha^2 s_1^2 s_2^2 e^4,$$

where

$$\Lambda_{n,i} = c_1^2 + c_2^2(-1)^i n c_1^2 c_2^2 (-1)^i.$$

As the van der Waals system includes several integrable cases, in order to study stability in those cases, higher order normalization, at least second order, is needed. This subject will be tackled in future works.

Here we try only to identify *relative equilibria*. In fact, this is the procedure to follow in order to classify and compute different types of orbits defined by our system. For the generic case, first order normalization is sufficient. The second order normalization is required when we consider degenerate situations, connected with integrable cases. For this reason we give both, first and second order in this section.

5 Searching for relative equilibria and invariant tori of the first order normalized system

Focusing on the task that we have announced before, we use the first order normalization of the Hamiltonian. Thus the corresponding differential system is given by

$$\dot{l} = \frac{\partial \mathcal{H}}{\partial L} = \frac{\gamma^2}{L^3} + \varepsilon \frac{\partial \mathcal{P}}{\partial L}, \quad \dot{L} = -\frac{\partial \mathcal{H}}{\partial l} = 0, \tag{35}$$

$$\dot{g} = \frac{\partial \mathcal{H}}{\partial G} = \varepsilon \frac{\partial \mathcal{P}}{\partial G}, \quad \dot{G} = -\frac{\partial \mathcal{H}}{\partial g} = 0, \tag{36}$$

$$\dot{u}_1 = \frac{\partial \mathcal{H}}{\partial U_1} = \varepsilon \frac{\partial \mathcal{P}}{\partial U_1}, \quad \dot{U}_1 = -\frac{\partial \mathcal{H}}{\partial u_1} = 0 \tag{37}$$

$$\dot{u}_3 = \frac{\partial \mathcal{H}}{\partial U_3} = \varepsilon \frac{\partial \mathcal{P}}{\partial U_3}, \quad \dot{U}_3 = -\frac{\partial \mathcal{H}}{\partial u_3} = 0 \tag{38}$$

In other words L, U_1 and U_3 are integrals, as we already know, and the system splits into a 1-DOF Hamiltonian system, namely

$$\dot{g} = \varepsilon \frac{\partial \mathcal{P}}{\partial G}, \quad \dot{G} = -\varepsilon \frac{\partial \mathcal{P}}{\partial g} \tag{39}$$

and three quadratures coming from

$$\dot{l} = \frac{\gamma^2}{L^3} + \varepsilon \frac{\partial \mathcal{P}}{\partial L}, \quad \dot{u}_1 = \varepsilon \frac{\partial \mathcal{P}}{\partial U_1}, \quad \dot{u}_3 = \varepsilon \frac{\partial \mathcal{P}}{\partial U_3},$$

as soon as the system Eqs. (39) is solved. We are not going to explore this way, which moreover involves hyperelliptic integrals. In other words, it will be more convenient to rely on numerical integrations.

Our previous work on this model concentrated on searching for relative equilibria related to singular points of the energy-moment map. Furthermore, those singular points correspond to several kind of motion that are not covered by the symplectic chart. Therefore, a complete analysis should be done using invariants rather than symplectic variables. In fact the lower dimensional relative equilibria, i.e. those that correspond to invariant \mathbb{S}^1 or \mathbb{T}^2 , are given by the singularity of the moment map for the \mathbb{T}^3 -symmetry group, and can be described by a moment polytope; for details see [18].

Excluding those solutions, in the remaining open domain we may use symplectic charts. Relative equilibria of the system (35) - (38) may be classified as invariant k -tori ($k = 1, 2, 3$), since our 4-DOF original system is endowed with two symmetries and the passage to a 1-DOF system is done by means of just one truncation process. Notice that the procedure we follow for obtaining periodic solutions and invariant tori is not valid for arbitrary perturbations. As a consequence of the Reeb Theorem [35, 37] one can only conclude the existence of periodic solutions as single non-degenerate relative equilibria and their associated linear stability.

• **invariant 3-tori.** They are the solutions (g, G) of the system defined by

$$\dot{G} = -\varepsilon \frac{\partial \mathcal{P}}{\partial g} = 0, \quad \dot{g} = \varepsilon \frac{\partial \mathcal{P}}{\partial G} = 0. \tag{40}$$

as functions of U_1, U_3 and β .

• **invariant 2-tori.** In a similar way, the search for *invariant 2-tori* splits into the search the roots of the subsystems. Namely

$$\dot{G} = -\varepsilon \frac{\partial \mathcal{P}}{\partial g} = 0, \quad \dot{g} = \varepsilon \frac{\partial \mathcal{P}}{\partial G} = 0, \quad \dot{u}_1 = \varepsilon \frac{\partial \mathcal{P}}{\partial U_1} = 0, \tag{41}$$

where the possible roots will be function of U_3 and β , and

$$\dot{G} = -\varepsilon \frac{\partial \mathcal{P}}{\partial g} = 0, \quad \dot{g} = \varepsilon \frac{\partial \mathcal{P}}{\partial G} = 0, \quad \dot{u}_3 = \varepsilon \frac{\partial \mathcal{P}}{\partial U_3} = 0, \tag{42}$$

where the possible roots will be function of U_1 and β .

• **periodic orbits:** The research for *periodic orbits* of the original system is equivalent [37] to find the roots of the first reduced system of equations

$$\dot{G} = -\varepsilon \frac{\partial \mathcal{P}}{\partial g} = 0, \quad \dot{g} = \varepsilon \frac{\partial \mathcal{P}}{\partial G} = 0, \quad \dot{u}_1 = \varepsilon \frac{\partial \mathcal{P}}{\partial U_1} = 0, \quad \dot{u}_3 = \varepsilon \frac{\partial \mathcal{P}}{\partial U_3} = 0, \tag{43}$$

Solutions of the system given by Eqs. (40) define invariant 3-tori $\mathbb{T}^3(\ell, u_1, u_3)$, except for the singular points [18]. The system defined by Eqs. (41) gives the invariant 2-tori $\mathbb{T}^2(\ell, u_3)$, each of them is attached to a 2-torus $\mathbb{T}^2(g, u_1)$ made of fixed points. Likewise the system defined by Eqs. (42) is the invariant 2-tori $\mathbb{T}^2(\ell, u_1)$, each of them is attached to a 2-torus $\mathbb{T}^2(g, u_3)$ made of fixed points. Finally, the intersection of the previous types of 2-tori, defined by the system of Eqs. (43), correspond to periodic solutions $S^1(\ell)$, each of them is attached to a 3-torus $\mathbb{T}^3(g, u_1, u_3)$ made of fixed points. They correspond to the short period solutions related to the unperturbed system. In what follows we search for invariant 3-tori and periodic solutions of short period.

5.1 Searching for invariant 3-tori

To find families of relative equilibria (periodic orbits, invariant tori, etc), we start searching for invariant 3-tori, the common condition for all the cases. Explicitly, according to above paragraphs, the equations (40) are

$$\frac{\partial C_{01}}{\partial G} + \frac{\partial C_{11}}{\partial G} \cos g + \frac{\partial C_{21}}{\partial G} \cos 2g = 0, \tag{44}$$

$$(C_{11} + 4C_{21} \cos g) \sin g = 0. \tag{45}$$

Due to the structure of Eq. (45), and keeping in mind the domain of existence of 4-D Delaunay variables, the only possibility leading to roots is $\sin g = 0$. Replacing in (44), we have an equation to be solved for each of the values of $\cos g$, namely ± 1 . Such equations are:

$$\frac{\partial C_{01}}{\partial G} + \frac{\partial C_{11}}{\partial G} + \frac{\partial C_{21}}{\partial G} = 0, \quad \frac{\partial C_{01}}{\partial G} - \frac{\partial C_{11}}{\partial G} + \frac{\partial C_{21}}{\partial G} = 0. \tag{46}$$

Both equations can be solved at one time applying the following strategy. By using the state functions (23) and (26), both equations (46) can be rewritten in the form; $R(\eta, z, w)x(\eta, z, w) + Q(\eta, z, w) = 0$ and $R(\eta, z, w)x(\eta, z, w) - Q(\eta, z, w) = 0$, in which:

$$\begin{aligned} x(\eta, z, w) &= \sqrt{(1-w^2)(1-z^2)}e, \\ R(\eta, z, w) &= -(3+4\alpha)\eta^6 + (5w^2+7w^2z^2+5z^2)\alpha\eta^2 - 20w^2z^2\alpha\eta^2, \\ Q(\eta, z, w) &= \alpha w z [\eta^8 - (10+11w^2+w^2z^2)\eta^4 - 11z^3\eta^3 \\ &\quad + (15w^2+17w^2z^2+15z^2)\eta^2 + 5z^5\eta - 20w^2z^2]. \end{aligned} \tag{47}$$

By multiplying equations $Rx + Q = 0$ and $Rx - Q = 0$ we obtain a new polynomial $P(\eta) = R^2x^2 - Q^2$ in η as independent variable, where w , z and α are taken as parameters. The roots of P represent the set of solutions for the original equations (46). Explicitly the polynomial is:

$$P(\eta) = a_6\eta^6 + a_5\eta^5 + a_4\eta^4 + a_3\eta^3 + a_2\eta^2 + a_1\eta + a_0, \quad (48)$$

where the coefficients are given by

$$\begin{aligned} a_6 &= -9 - 24\alpha - 16\alpha^2, \\ a_5 &= w^2(16\alpha^2 + 9 - \alpha^2z^2 + 24\alpha) + 9 + 24\alpha + 16\alpha^2 + z^2(24\alpha + 9 + 16\alpha^2), \\ a_4 &= w^2(24\alpha^2 + 18\alpha z^2 + 6\alpha - 9 - 9z^2 + 30\alpha^2z^2w^2) + z^2(6\alpha + 24\alpha^2 - 9), \\ a_3 &= w^4(-40\alpha^2 - 42\alpha z^2 - 30\alpha - 34\alpha^2z^2 + 2z^4) + 9z^2w^2 - 30z^2 - 40\alpha^2z^2 \\ &\quad - 40\alpha^2z^4 + \alpha w^2(198z^2 - 34\alpha z^4 + 30 + 285\alpha z^2 + 40\alpha + 42z^4) - 30\alpha z^4, \\ a_2 &= \alpha(30w^4 + 42z^4w^4 + 192z^2w^4 + 15\alpha w^4 - 17\alpha z^4w^4 + 266\alpha z^2w^4 - 30z^4 \\ &\quad + 266\alpha z^4w^2 + 192z^4w^2 + 180z^2w^2 + 290\alpha z^2w^2 + 15\alpha z^4), \\ a_1 &= \alpha^2(25w^4 - w^6z^6 + 27w^6z^4 - 51w^6z^2 + 25w^6 + 27z^6w^4 - 139z^4w^4 \\ &\quad - 225z^2w^4 + 25z^6 + 25z^4 - 51z^6w^2 - 225z^4w^2 - 50z^2w^2) \\ &\quad + 150z^2w^4 + 150z^4w^2 + 162z^4w^4, \\ a_0 &= 5\alpha(\alpha w^6z^4 - 3\alpha w^6z^6 - 5\alpha w^6 + 7\alpha w^6z^2 + 24z^4w^4 + \alpha z^6w^4 \\ &\quad + 5\alpha z^2w^4 + 18\alpha z^4w^4 - 5\alpha z^6 + 5\alpha z^4w^2 + 7\alpha z^6w^2). \end{aligned}$$

This polynomial corresponds with the one obtained from the system (14) when we look for relative equilibria within the Poisson approach [18]. The complexity in both analyses is similar because the polynomials are of the same degree. It is straightforward to check that when we impose the constraint $z = 0$, we recover the expressions of the study done with the classical Delaunay variables [22].

The search for possible invariant \mathbb{T}^2 solutions of Systems (41) and (42) requires a similar analysis. We will not refer to them here. Instead, we focus on periodic orbits, where we obtain a benefit working in symplectic formalism.

5.2 Searching for periodic orbits

With Poisson formalism, finding periodic solutions is in correspondence with computing relative equilibria of the system defined with the normalized Hamiltonian (9). This study requires to deal with the constraints defining $\mathbb{C}\mathbb{P}^3$ and leads to a large polynomial system; for details see [18]. Although partial results are obtained, the presence of the physical parameter introduces complicated expressions.

The situation changes dramatically when we switch to symplectic formalism. The complete system (43) related to periodic solutions takes the form

$$\begin{aligned} \frac{\partial C_{01}}{\partial G} + \frac{\partial C_{11}}{\partial G} \cos g + \frac{\partial C_{21}}{\partial G} \cos 2g &= 0, \\ (C_{11} + 4C_{21} \cos g) \sin g &= 0, \\ \frac{\partial C_{01}}{\partial U_1} + \frac{\partial C_{11}}{\partial U_1} \cos g + \frac{\partial C_{21}}{\partial U_1} \cos 2g &= 0, \\ \frac{\partial C_{01}}{\partial U_3} + \frac{\partial C_{11}}{\partial U_3} \cos g + \frac{\partial C_{21}}{\partial U_3} \cos 2g &= 0. \end{aligned}$$

The solutions of the system are the set of the periodic orbits in the first order normalized problem. By solving

the above system imposing $\alpha \neq 0$, we obtain that periodic orbits are characterized by $U_1 = U_3 = 0$ and

$$\begin{aligned} |U_1| = |U_3| &= \sqrt{\frac{e \cos g (4 + 5e \cos g + e^2) + 1 - e^2}{e \cos g (5e \cos g + 8 + 2e^2) + 3 + 2e^2}} \\ |U_1| = |U_3| &= \sqrt{\frac{e \cos g (-4 + 5e \cos g - e^2) + 1 - e^2}{e \cos g (5e \cos g - 8 - 2e^2) + 3 + 2e^2}} \end{aligned} \tag{49}$$

Then (43) becomes a system of three equations given by

$$\begin{aligned} \frac{\partial C_{01}}{\partial G} + \frac{\partial C_{11}}{\partial G} \cos g + \frac{\partial C_{21}}{\partial G} \cos 2g &= 0, \\ (C_{11} + 4C_{21} \cos g) \sin g &= 0, \\ \frac{\partial C_{01}}{\partial U_3} + \frac{\partial C_{11}}{\partial U_3} \cos g + \frac{\partial C_{21}}{\partial U_3} \cos 2g &= 0. \end{aligned} \tag{50}$$

Again we look for solutions when $\sin g = 0$. Then, we drop the second equation and we have two systems to solve related to $\cos g = \pm 1$:

$$(i) \quad \frac{\partial C_{01}}{\partial G} + \frac{\partial C_{11}}{\partial G} + \frac{\partial C_{21}}{\partial G} = 0, \quad \frac{\partial C_{01}}{\partial U_3} + \frac{\partial C_{11}}{\partial U_3} + \frac{\partial C_{21}}{\partial U_3} = 0, \tag{51}$$

and

$$(ii) \quad \frac{\partial C_{01}}{\partial G} - \frac{\partial C_{11}}{\partial G} + \frac{\partial C_{21}}{\partial G} = 0, \quad \frac{\partial C_{01}}{\partial U_3} - \frac{\partial C_{11}}{\partial U_3} + \frac{\partial C_{21}}{\partial U_3} = 0. \tag{52}$$

Considering case (i), from the second equation we obtain

$$c_2 (e^2 + 3e + 1 - c_2^2 (2e + 3)(e + 1)) = 0.$$

Thus, we have either $c_2 = 0$ or

$$c_2^2 = \frac{1 + 3e + e^2}{(3 + 2e)(1 + e)}. \tag{53}$$

If $c_2 = 0$, we obtain $\alpha = -3/4$. Otherwise, from the first equation we find

$$\alpha = \frac{3e(3 + 2e)^2}{3e^4 + 2e^3 - 10e^2 - 3e + 8} \tag{54}$$

Likewise, for the case (ii) the equation is

$$c_2 (e^2 - 3e + 1 - c_2^2 (2e - 3)(e - 1)) = 0.$$

Hence, again $c_2 = 0$ or

$$c_2^2 = \frac{1 - 3e + e^2}{(2e - 3)(e - 1)}. \tag{55}$$

If $c_2 = 0$, we obtain $\alpha = -3/4$. Otherwise, from the first equation now we find

$$\alpha = -\frac{3e(3 - 2e)^2}{3e^4 - 2e^3 - 10e^2 + 3e + 8}. \tag{56}$$

Thus, for each member of the van der Waals family defined by α , we have the relation between integrals which give periodic orbits. The period is computed replacing those values in the right hand terms of (35).

Conclusions and future work

In contrast to some claims in [7], Poisson and symplectic formulations of perturbed 4-D isotropic oscillators systems behave as complementary in the analysis of the main features of those systems. The first approach, built on the invariants and their algebraic relations, is necessary in order to carry out regular and singular reductions of the systems, in particular when singular points are present in the reduced spaces. Nevertheless the use of those invariants and their relations, introduce large computations in the search of relative equilibria, specially when physical parameters are involved. Only those equilibria related to the symmetry groups of the system are more easily computable.

The situation is rather different when the symplectic approach is followed. From the geometric mechanics point of view this helps to portrait the toral structure of the phase space, necessary when stability KAM theory is applied. Moreover, from the algebraic perspective, the number of equations and relations involved falls sharply because there are no constraints. Normal forms are computed very efficiently in this frame. Of course, this is at the expense of considering only an open domain of the phase space, where those symplectic variables are defined. For this reason we should begin with Poisson formalism. It is only after studying singular points that we can switch to symplectic techniques with properly chosen variables.

As an illustration, the van der Waals family is studied in detail here, showing the pros and cons of both approaches. The reconstruction process connecting the whole analysis of relative equilibria with the original system, is still to be tackled. In particular, the analysis of the different types periodic orbits.

An open question is whether there are other invariants which can lead to similar equations to those appearing in symplectic variables. In the same vein, connected with our choice as 4D-Delaunay variables [24, 27], it might be worth considering other recent proposals [33, 34].

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