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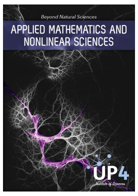
Applied Mathematics and Nonlinear Sciences 1(2) (2016) 391–404



Applied Mathematics and Nonlinear Sciences

http://journals.up4sciences.org

On problems of Topological Dynamics in non-autonomous discrete systems

Francisco Balibrea[†].

Departamento de Matemáticas, Campus de Espinardo, Universidad de Murcia, 30100 Murcia
SPAIN

Submission Info

Communicated by Juan L.G. Guirao
Received 7th January 2016
Accepted 28th September 2016
Available 28th September 2016

Abstract

Most of problems in Topological Dynamics in the theory of general autonomous discrete dynamical systems have been addressed in the non-autonomous setting. In this paper we will review some of them, giving references and stating open questions.

Keywords: non-autonomous systems; topological entropy; Li-Yorke chaos; minimality; forbidden sets; Lyapunov exponents
AMS 2010 codes: 37B55.

1 Introduction, definitions and first results

The interest for non-autonomous discrete systems *(n.a.d.s.)* or simply *(na)* has been increasing in last years because they are adequate to model some phenomena in applied sciences, such as biology [32,53], physics [38], economy [55], etc., and to solve problems generated in mathematics (see [41]).

By other hand, more realistic models in the setting of dynamical systems are those where the trajectories of all points in the phase state are affected by small random perturbations. Most of such situations can be studied following the methodology of non-autonomous systems. In the autonomous case, we have a phase space and a unique continuous map where the trajectories of points are obtained iterating such map. For non-autonomous systems, the trajectories are produced using iteration methods by changing the map in each step.

Keeping the above ideas in mind, we are introducing precisely the general setting of *(na)*. Let $(X_i)_{i=0}^{\infty} = X_{0,\infty}$ be a sequence of Hausdorff topological spaces and $(f_i)_{i=0}^{\infty} = f_{0,\infty}$ a sequence of continuous maps, where $f_i : X_i \to X_{i+1}$ for $i \in \mathbb{N}^{\star} = \mathbb{N}\bigcup\{0\}$. For any pair of positive integers (i,n), we set

$$f_i^n = f_{i+(n-1)} \circ f_{i+(n-2)} \circ \cdots \circ f_{i+1} \circ f_i.$$

[†]Corresponding author.
Email address: balibrea@um.es

ISSN 2444-8656 doi: 10.21042/AMNS.2016.2.00034

We also state $f_i^0 = \text{Identity}|_{X_i}$ and $f_i^{-n} = (f_i^n)^{-1}$ (taken in the sense of inverse images when the maps are not invertible).

The pair $(X_{0,\infty}, f_{0,\infty})$ is a *(na)* in which the sequence $(f_1^n(x))_{n=0}^{\infty} = (x_n)_{n=0}^{\infty}$, where $x_0 = x$ is the *trajectory of the point* $x \in X_0$. The set of points of that trajectory is the *orbit of* $x \in X_1$. In some problems, we will denote by $(X_{1,\infty}^{[n]}, f_{1,\infty}^{[n]})$ the $n-th$ *iterate* of the system, that is, $X_i^{[n]} = X_{(i-1)n+1}$ and $f_i^{[n]} = f_{(i-1)+1}^{[n]}$.

In order to have more concrete results, in particular in applications to real models in sciences, we will be restricted to the case when all spaces X_i are compact or compact and metric (in the last case, we will denote by $(d_n)_{n=1}^{\infty}$ the corresponding sequence of metrics). It is evident that when all spaces coincide with X and all maps with f, then we simply have the *autonomous discrete dynamical system* denoted by the pair (X, f).

In the literature on *(na)* there are a lot of results in the case when all spaces X_i are real compact intervals and the continuous maps f_i are of a particular type, for example, piecewise linear maps (see for example [44]).

Of interest is also the case when the spaces and maps, components of the non-autonomous system, fulfill some periodic conditions.

Definition 1. A *(na)* is $p-$*periodic* if $X_{n+p} = X_n$ and $f_{n+p}(x) = f_n(x)$ for $x \in X_n$, $n \geq 0$, and $p \geq 1$ being a positive integer. If $p = 1$, then we have the autonomous case.

Such non-autonomous systems have deserved special interest to many researchers in the theory of dynamical systems trying to extend to them the topics of Topological Dynamics considered in the autonomous case. For some of them, see [50] and the references therein.

In most cases, it is supposed we have only a topological space $X = X_i$ for all $n \in \mathbb{N}^\star$ but a sequence of distinct maps. The resulting system will be denoted by $(X, f_{0,\infty})$. At most all applications deal with this case. Moreover, in order to obtain more concrete results, we will take X as a compact metric space.
The rest of the paper will be devoted to the consideration of the well known topics on Topological Dynamics for autonomous systems but now applied to the non-autonomous setting.

The introduction of non-autonomous systems in mathematics has been motivated by the computation of the topological entropy for skew product or triangular discrete dynamical systems in the unit square $[0,1]^2 = I^2 = Q$, that is, discrete dynamical systems (Q, F), where $F(x,y) = (f(x), g(x,y)) = (f(x), g_x(y))$ and $F : Q \to Q$ is a continuous self-map (written $F \in C(Q)$). The notion of *(na)* was formulated in [41] as an extension of that of *autonomous system* and it was studied the topological entropy. The introduction was made considering the above notion of trajectory or orbit of points of the spaces. The extension of the notion of topological entropy can be made using *covers* in the way of [1] or using the *Bowen's formula* (see [23]).

$$\max\{h(f), h_f(F)\} \leq h(F) \leq h(f) + h_f(F),$$

where $h(F), h(f)$ denote respectively the topological entropy of F and f, $h_f = \sup\{h(F|I_x)\}$, and $I_x = \{x\} \times I$. But to compute $h_f(F)$ is necessary to obtain the trajectory of every point $(x,y) \in I_x$, and this implies the knowledge of the sequence $(g_x(y), g_{f(x)}(y), \ldots, g_{f^n(x)}(y), \ldots)$, that is, in every step of the iteration, the map to be taken is different and must follow the sequence of maps (Identity on $I, g_x, g_{f(x)}, \ldots, g_{f^n(x)}, \ldots$).

S. Kolyada and L'. Snoha introduced in [41] the notion of topological entropy for *(na)* adapting the original definition of [1], denoted by $h(f_{0,\infty})$, using covers, and when X is metrizable that one in [23], using *spanning and separating sets*. For compact metric spaces both notions are the same. If X is a compact topological space, then it is proved that $h(f_{0,\infty}^n) \leq n \cdot h(f_{0,\infty})$ for every $n \geq 1$, where $f_{0,\infty}^n = (f_{in+1}^{\infty})_{i=0}^{\infty}$. When $f_{0,\infty}$ is periodic of period p, ($f_{n+p} = f_n$ for every n) then $h(f_{0,\infty}^p) = p \cdot h(f_{0,\infty})$. If X is a compact metric space and $f_{0,\infty}$ is a sequence of *equicontinuous* self-maps, then $h(f_{0,\infty}^n) = n \cdot h(f_{0,\infty})$ for every $n \geq 1$. In [41] it was also introduced the notion of *asymptotically topological entropy*, $h^\star(f_{0,\infty})$, as $\lim_{n\to\infty} h(f_{n,\infty})$ where $f_{n,\infty}$ is the tail from n of the sequence $f_{0,\infty}$. It is proved that such a limit always exists. Many results on usual topological entropy of autonomous systems are held by the asymptotically topological entropy, proving that in different settings the two notions are analogous. In [41], it was also proved that for compact metric spaces, if the sequence $f_{0,\infty}$ converges uniformly to f or it is an equicontinuous family, then $h^\star(f) \leq h(f)$.

One relevant consequence of the entropy in the non-autonomous case is the proof that in autonomous cases, the topological entropy is commutative for the composition of two continuous maps in compact metric spaces, that is, $h(f\circ g) = h(g\circ f)$ for $f,g\in C(X)$. Additionally, in [41] some other results were proved.

2 Topological entropy

We introduce the notion of topological entropy in the setting of non-autonomous systems of the form $(X, f_{0,\infty})$, where (X,d) is a compact metric space. We follow the Bowen's line of introduction of the notion and the notation above considered and also [15].

For $x,y\in X$ and $n\geq 0$,

$$\rho_n(x,y) = \max_{i=0,\ldots,n-1} d(f_0^i(x), f_0^i(y)).$$

The set $E\subset X$ is said to be $(n,\varepsilon,f_{0,\infty})-$*separate* if $\rho_n(x,y)>\varepsilon$ for every distinct $x,y\in E$. Now denote by $s_n(f_{0,\infty})$ the maximal cardinality of $(n,\varepsilon,f_{0,\infty})-$*separate* sets. Then the topological entropy of $(X,f_{0,\infty})$ is

$$h(f_{0,\infty}) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log(s_n(f_{0,\infty},\varepsilon)).$$

This definition is just an extension of the topological entropy for autonomous systems in compact metric spaces.

2.1 Topological entropy and limits

We state the question of what is the behaviour of the entropy of a *(na)* given by the pair $(X,f_{0,\infty})$ if the sequence $(f_n)_{n=1}^{\infty}$ converges to a continuous map $f\in C(X)$. We will consider uniform or piecewise convergences. They are expected different behaviours in these cases.

In the next result, proved in [41], we consider the case where the convergence is uniform.

Theorem 1. *Let X be a compact metric space and $(f_i)_{i=0}^{\infty}$ be a sequence of continuous maps converging uniformly to a continuous map $f\in C(X)$. Then $h(f_{0,\infty})\leq h(f)$.*

In the following examples we see that if the convergence of the sequence is piecewise to f but not uniform, the above statement is not true in general and it is possible to contruct some examples. In the case that $h(f)=\infty$, the previous formula is true. In [15], we proved the following result.

Proposition 2. *For every continuous interval map f, there is a non-autonomous system $(I,f_{0,\infty})$ such that $(f_n)_{n=1}^{\infty}$ converges pointwise to a continuous map f and $h(f_{0,\infty})=\infty$.*

Proof. Choose in $[0,1]$ an infinite sequence of closed intervals $[a_n,b_n]$, for example, take $a_n=\frac{1}{2^n}$ and $b_n=a_n+\frac{1}{4^n}$. From this, $a_n-b_{n+1}=\frac{1}{2^{n+1}}-\frac{1}{4^{n+1}}>0$ for all $n\in\mathbb{N}$, which means that the above election is possible. Then in each interval $[a_n,b_n]$ we choose $n-$*subintervals* $[c_k,d_k]$ from $k=1,\ldots,n$ with $d_k=a_k+\frac{1}{4^{n+k-1}}$ and $c_n^k=d_n^{k+1}$. Then inside every subinterval $I_n^k=[c_n^k,d_n^k]$, we choose another subinterval $[\alpha_n^k,\beta_n^k]$ taking $\alpha_n^k=c_n^k+\frac{1}{10}(d_n^k-c_n^k)$ and $\beta_n^k=c_n^k+\frac{9}{10}(d_n^k-c_n^k)$.

Given $f\in C(I)$, for every $n\in\mathbb{N}$ we define $f_n(x)=f(x)$ for all $x\notin\bigcup_{k=1}^{n}(I_n^k)$, $f_n(c_n^k)=f(c_n^k)$, $f_n(d_n^k)=f(d_n^k)$, $f_n(\alpha_n^k)=1$, $f_n(\beta_n^k)=0$, and in the rest of I_n we define f_n connecting the dots.

In fact, what has been done is to introduce in every subinterval I_n^k a linear perturbation in such a way that f_n results continuous and $f_n(I_n)=I$ for every n, that is, f_n is surjective.

First, it is evident that in I, $\lim_{n\to\infty}f_n(x)=f(x)$ is point-wisely since the perturbation is acting on I_n only for the index n but not for the rest of indexed of the limit. Besides, the perturbation is moving to the left when n increases.

Now consider a fixed m and any $n>m$. It is evident that $f_m(I_m)\subset\bigcup_{k=1}^{n}I_k$ creating an infinite number of *horseshoes*. As a consequence, by applying Theorem 3 of [2], we have $h(f_{1,\infty})\geq\log m$. But since m is arbitrarily large, then we conclude that $h(f_{1,\infty})=\infty$.

Remark 1. Using the above result and construction, choosing the continuous map f with zero, positive or infinite topology, we have examples of three types with $h(f_{0,\infty}) = \infty$.

Open Question 1. *Let $(f_n)_{n=1}^{\infty}$ be pointwise convergent to f.*

1. *Construct an example of a non-autonomous system for which $h(f_{0,\infty}) > 0$ and $h(f) = 0$.*
2. *Construct an example of a non-autonomous system such that $h(f_{0,\infty}) = 0$ and $h(f) = 0$.*

2.2 Topological entropy and Li-Yorke chaos

Using the definition of trajectories for non-autonomous systems, next we state the defintion of Li-Yorke chaos for *(na)* in the same sense as in the autonomous case.

Definition 2. Let $(X, f_{0,\infty})$ and $x, y \in X$. The pair (x, y) is *Li-Yorke chaotic* if

1. $\liminf_{n\to\infty} d(f_1^n, f_1^n) = 0$, and
2. $\limsup_{n\to\infty} d(f_1^n, f_1^n) > 0$.

Definition 3. A set $S \subset X$ is a *scrambled set* if it is uncountable and every pair of distinct points $x, y \in S$ is Li-Yorke chaotic.

Definition 4. $f \in C(X, X)$ is *Li-Yorke chaotic* if it possesses an uncountable scrambled set.

For autonomous systems in compact metric spaces, it is proved in [18] that positive topological entropy implies the existence of Li-Yorke chaos. In the following result from [15], it is proved that in general it is not true in *(na)*. This was proved by constructing an interval example composed only by two different maps such that $h(f_{0,\infty}) > 0$ and the sequence $(f_i)_{i=1}^{\infty}$ converges to a map f which is not Li-Yorke chaotic.

Open Question 2. *Using the methodology and approaches of [37, 52], try to extend the results of these papers to general metric spaces.*

Theorem 3. *There exists a (na) on the interval, $(I, f_{0,\infty})$, such that the sequence $(f_i)_{i=0}^{\infty}$ converges pointwisely to a non-continuous map $f$and satisfying that*

1. *$h(f_{0,\infty}) > 0$.*
2. *f is not Li-Yorke chaotic.*

Proof. According to [15], take the interval $[0,1]$ and divide it into three subintervals of length $\frac{1}{3}$. Denote the central subinterval by J and consider the two piecewise linear maps f_1 and f_2 (see [15]). Consider now the sequence of maps composed by f_1 and f_2 where the map f_1 appears infinitely many times. With such a distribution, the points $0, 1$ are fixed and the rest of points of $[0,1]$ are asymptotic to 0. As a consequence, the pointwise limit of the initial sequence is a non-continuous map.

The behavior of g in the sequence $f_{0,\infty}$ is described as follows. Take $m_0 = 1$, $m_n = 2^{n^2}$, and put $f_n = g$ for $n = m_n$ and $f_i = h$ for any other index. The autonomous system $([0,1], h)$ has a $2-$*strong horseshoe* in the subinterval J. Then there is ε such that for every n, there exists an $(n, \varepsilon, h)-$separate set $E \subset J$ holding $\mathrm{card}(E) = 2^n$.

For every n there is an interval K_n such that $f_1^{m_n}(K_n) = J$. We state $K = g^{-n-1}(J)$, let $l_n = m_{n+1} - m_n - 1$, and F_n be an $(l_n, \varepsilon, h)-$separate set of h having maximal cardinality. Then $K_n = g^{-n-1}(F_n) \subset K$ is $(m_{n+1}, \varepsilon, f_{0,\infty})-$separate. That is, for $m_n \le j < m_{n+1}$, we have $f_1^j(K_n) = g_{m_n}^{j-m_n} = h^{j-m_n}(F_n)$. Then

$$\begin{aligned} h(f_{0,\infty}) &\ge \limsup_{n\to\infty} \frac{1}{m_{n+1}-1} \log \mathrm{card}\, F_n \ge \lim_{n\to\infty} \frac{l_n}{m_{n+1}} \log 2 \\ &\ge \lim_{n\to\infty} \frac{m_{n+1}-m_n-1}{m_{m+1}} \log 2 \ge \left(1 - \lim_{n\to\infty} 2^{(n^2+1)-(n+1)^2}\right) \log 2 \ge \log 2. \end{aligned}$$

Open Question 3. *1. Prove that for a (na) system of the form of the form $(I, f_{0,\infty})$ composed by onto maps converging uniformly to f, it holds that $h(f) = h(f_{0,\infty})$.*

2. In other spaces different from the interval I, construct examples of onto continuous maps f_n converging uniformly to f and such that

(a) $h(f) = \infty$ and $h(f_{0,\infty}) = \infty$.

(b) $h(f) = \infty$ and $h(f_{0,\infty}) > 0$.

(c) $h(f) = \infty$ and $h(f_{0,\infty}) = 0$.

Similar results to above have been obtained in [52] for $(I^m, f_{0,\infty})$ where the sequence of maps converges uniformly to a map in I^m and all trajectories are subjected to small random perturbations. In fact, it is proved that if f is the limit map of the sequence of maps and $P \in I^m$ is *recurrent* in the autonomous dynamical system (I^m, f), then P is also recurrent in the non-autonomous case (see definitions of recurrence in [34]) affected by small random perturbatons. It is also proved that under some sufficient conditions, a non-autonomous system $(I, f_{0,\infty})$ subjected to small perturbations can be non-chaotically converted in the Li-Yorke sense.

2.3 Minimal sets

We say that an autonomous system (X, f) is *minimal* if there is no proper subset $M \subset X$ which is non-empty, closed, and $f-$invariant $(f(M) \subseteq M)$. Then we also say that the map f is *minimal*. It is immediate that f is minimal, if and only if, the forward orbit of every point $x \in X$ is dense on X (see [34]).

We say that $(X, f_{1,\infty})$ is *minimal* if every trajectory is dense in X. Some properties of minimal autonomous systems, such as f being *feebly open* (the map transforms open sets into sets with non-empty interiors) or *almost one-to-one* (a typical point has just one pre-image) are not held in the setting of non-autonomous systems (see [43]). For example, the former properties cannot be obtained neither for the maps f_n nor $f_n \circ f_{n-1} \circ \cdots \circ f_1 \circ f_0$. But this is not the unique property: in fact, there is a wider variety of dynamical behaviours in the non-autonomous systems than in the autonomous cases.

In [42] it is proved that it is equivalent for $(X, f_{0,\infty})$ not being minimal to the fact that there is a non-empty open set $B \subseteq X$ such that the system has arbitrarily long finite trajectories disjoint with B. This has as a corollary a sufficient condition for metric spaces without isolated points to be non-minimal. That condition holds whether there is a nonempty open set $B \subseteq X$ and $n_0 \in \mathbb{N}$ satisfying the two following:

1. $f_1^{n_0-1}$ is onto.
2. The non-autonomous system has arbitrarily long finite trajectories disjoint with B.

The same happens with the conditions $f_1^{n_0-1}$ as well as the maps f_n for $n \geq n_0$ which are onto and for every $n \geq n_0$, $f_n(B) \subseteq f_n(X \backslash B)$. Under the former conditions for $(X, f_{1,\infty})$, suppose that the sequence $(f_n)_{n=1}^{\infty}$ converges uniformly to f. If f is not onto, then the system is not minimal, and even more, no trajectory is dense.

In [42], there is a discussion using examples in $X = [0,1]$ to check the validity of the former conditions and to prove that even if $f_n \to f$ and for every $n \in \mathbb{N}$, f_n is onto, then f is only monotonic (not necessarily strict). Theorem 3.2 in [42] proves the existence of $(I, f_{0,\infty})$ such that f_n converges uniformly to the Identity on I, for every n, f_n is onto and can be chosen piecewise linear with non-zero slopes, and such that $(I, f_{n,\infty})$ is a minimal system. The arguments stated in the referred results are used to construct and improve some examples introduced in [13] in the setting of skew product maps on Q with the property that almost all orbits in Q have the second projection dense in I and whose omega-limit sets are $\{0\} \times I$.

2.4 Topological entropy of non-autonomous systems on the square and on $\mathbb{R}^2$

We have remarked previously that the computation of topological entropy in triangular systems on the square given by $F(x,y) = (f(x), g_x(y))$ are related to the consideration of trajectories of points $y \in [0,1]$, given by

$(g_x(y), g_{f(x)}(y), \ldots, g_{f^n(x)}(y), \ldots)$. As a consequence, some results on entropy of autonomous triangular systems can be obtained from non-autonomous systems as defined above. In [44], it is developed a theory of the topological entropy for non-autonomous piecewise monotonic systems on the interval. It is made with additonal conditions in the system, namely, for systems $(I_{1,\infty}, f_{0,\infty})$ being *bounded and long-lapped* or *Markov* (see [44] for such notions). In the next result, we denote by $c_{1,n}$ the number of pieces of monotonocity of the map $f_{1,n}$.

Theorem 4. *Let $(I, f_{0,\infty})$ be finite piecewise monotonic or Markov. Then the non-autonomous system satisfies*

$$h(f_{0,\infty}) = \limsup_{n\to\infty} \frac{1}{n} \log c_{1,n}.$$

As an application of that result, in [44] it is constructed a large class of triangular maps on the square of type 2^∞ (such maps have periodic trajectories of all periods, all being powers of two) of class C^∞, extending a previous result appeared in [12].

Open Question 4.

1. *Prove or disprove if those triangular maps may be obtained in the class of real analytic or polynomial maps.*
2. *In case of a negative answer, construct adequate counterexamples.*

With respect to minimality, in [42] it is proved the existence of minimal non-autonomous systems on the interval, $(I, f_{0,\infty})$, such that the sequence $(f_n)_{n=0}^\infty$ converges uniformly to the *identity map* and all maps f_n are onto. Even more, all f_n can be choosen piecewise linear in I with non-zero slopes, having at most three pieces of linearity, and for every n, the *(na)*, $(I, f_{n,\infty})$ being minimal.

Such results are used to prove a result on autonomous triangular systems on the square (see [13, 42]).

Theorem 5. *There is a triangular map F defined on the square I^2 satisfying that*

1. *All points of the form $(0,y)$ are fixed.*
2. $\lim_{n\to\infty} f^n(x) = 0$ *for every x.*
3. *Every point in I^2 not being of the form $(0,y)$ has a trajectory whose second projection is dense in I.*
4. $h(F) = 0$.

A relevant fact in the proof of this theorem [42] consists of using an Extension Lemma (see [35] or [40]) which allows to carry out adequate constructions and to obtain properties in subsets of I^2 which can be extended to the total square keeping the properties.

Open Question 5.

1. *Try to obtain an example of the previous theorem in the class C^∞.*
2. *Is there a triangular (na) on the square such that $(f_n)_{n=0}^\infty$ converges pointwise to f and $h(f_{0,\infty}) = 0$ but $h(f) > 0$?*

One of the most known general two-dimensional map defined in $\mathbb{R}^2$ is the *Hénon map* which is given by

$$H_{a,b}(x,y) = (a + by - x^2, x),$$

where a and b are real parameters. When $b \neq 0$, then the map has an inverse given by

$$H_{a,b}^{-1}(x,y) = \left(y, \frac{x - a + y^2}{b}\right).$$

If $b=0$, then we have essentially the map $H_a(x,y)=(a-x^2,x)$ which behaves similarly to a one dimensional map. Therefore, we will deal with the case with parameter $b\neq 0$. It can be proved that there exists a Cantor invariant set $K\subset\mathbb{R}^2$ $(H(K)=K)$ where the map is topologically conjugate to a *shift map* defined on a finite number of symbols. Therefore, in K the map is *Devaney chaotic* (a proof can be seen in [54]). That proof applies sufficient conditions called *Conley* and *Moser conditions* (see [47]).

In [16], it is considered a non-autonomous version of the Hénon map when $b=-1$. For this value, the map is *area preserving*. The other parameter is allowed to vary for each n by

$$f_n(x,y)=(a(n)-y-x^2,x)$$
$$f_n^{-1}=(y,a(n)-x-y^2).$$

The sequence $a(n)$ will be taken as a trigonometric perturbation of the number $\frac{19}{2}$, that is

$$a(n)=\frac{19}{2}+\varepsilon\cos(n),\quad \varepsilon=\frac{1}{10}.$$

The choosing of the two values are to be able to apply a new version of the Conley-Moser condition (see [16]). The domains D_n are introduced as

$$D_n=D=[-R,R]\times[-R,R]$$
$$R=\sqrt{1+a(0)}.$$

A similar approach has been recently applied in [46] to the Lozi map on $\mathbb{R}^2$.

Open Question 6.

1. *Solve the same problem of Hénon (na) for* $b\notin\{-1,0\}$*, that is when the system is not conservative.*
2. *Consider a perturbation for* $a(n)$ *of the form*

$$a(n)=a_0+\varepsilon\,\mathrm{cs}\,(n,m),$$

   where cs denotes the Jacobi cosam elliptic maps. Alternately, it can be done also using the Jacobi senam elliptic map. See all previous results and statements in [8].
3. *Solve the previous questions for Lozi system.*

3 Difference and systems of difference non-autonomous equations

3.1 Examples in difference equations

For a wide range of types of difference equations, autonomous or non-autonomous, deterministic or stochastic, discrete or continuous, it has been proved that the asymptotic distribution of trajectories hold very often the so called Benford's law, which we introduce in this subsection. Frequently numerical data got from dynamical systems follows such law.

Firstly, we are dealing with dynamical properties associated to *Benford sequences*. It is known (see [21]) that frequently, trajectories from discrete dynamical systems satisfy the *Benford's law of logarith mantissa distributions*. This law is the probability distribution of the *mantissa function* or simply *mantissa* with respect to a base $b\in\mathbb{N}\setminus\{1\}$. This is given by

$$\mathbb{P}(\text{mantissa}_b\le t)=\log_b t,\quad t\in[1,b).$$

The mantissa function, denoted by $<\cdot>$, is a function from $\mathbb{R}^+$ to $[1,b)$ given by $<x>_b = b^{[\log_b x]}$. With this, we state the following

Definition 5. A sequence $(x_n)_{n=0}^{\infty}$ of real numbers is $b-$Benford if

$$\lim_{n\to\infty} \frac{\operatorname{card}\{j<n : <|x_j|>_b \leq t\}}{n} = \log_b t, \quad t\in[1,b),$$

and it is strictly Benford if it is $b-$Benford for every $b\in\mathbb{N}\setminus\{1\}$.

It is well-known (see for example, those of [21] and other references therein) the following result which compares the Benford property of a sequence and the same for the sequence of the log of the absolute value of their terms.

Proposition 6. *A sequence of real numbers* $(x_n)_{n=0}^{\infty}$ *is B−Benford, if and only if, the sequence* $(\log|x_n|)_{n=0}^{\infty}$ *is uniformly distributed modulus* 1.

Using the above result and others from uniformly distribution, in many examples can be proved the property of trajectories of dynamical systems starting in an initial point x_0 or simply general sequences of real numbers.

Example 7.

1. *The sequence* $(n!)$ *is Benford.*
2. *The trajectories constructed from the Fibonacci equation* $F_{n+2} = F_n + F_{n+1}$ *for* $n\geq 0$ *are Bedford, except for the starting point* $(0,0)$.
3. *For almost Lebesgue initial point* $x_0\in\mathbb{R}$*, the corresponding trajectoy is Benford.*
4. *The sequence* $(2^n)_{n=0}^{\infty}$ *is Benford if* $\log_b 2$ *is irrational, that is, if and only if,* $b\neq 2^j$ *for some* $j\in\mathbb{N}$.
5. *Not all sequences are Benford. For example,* (n^{α}) *for* $\alpha\in\mathbb{R}^+$ *and the sequence of prime numbers are not Benford for any b.*

In the case of *(na)* there are a few results in the literature. We will refer here to those from [19] concerning non-autonomous linear systems

$$x_n = A_n x_{n-1}, \quad n\in\mathbb{N},$$

where for every n, A_n is a real $m\dot{m}-$-matrix, and where the problem is to study under what conditions the mantissa distribution generated for the trajectories with initial conditions $x_0\in\mathbb{R}^d$ satisfy the Benford law. The results we obtain are related to the *b-resonance condition* introduced in [19].

Definition 6.

1. A set $\Lambda\subset\mathbb{C}$ is $b-$resonant if there exists a finite non-empty subset $\Lambda_0 = \{\lambda_1,\ldots,\lambda_q\}\subset\Lambda$ with $|\lambda_1| = \cdots = |\lambda_q|$ such that either $\operatorname{card}(\Lambda_0\cap\mathbb{R}) = 2$ or the numbers $1, \log_b|\lambda_1|$ as well as the elements of $\{\frac{1}{2\pi}\arg\lambda_1,\ldots,\frac{1}{2\pi}\arg\lambda_q\}\setminus\{0,\frac{1}{2\pi}\}$ are $\mathbb{Q}-$dependent.
2. The matrix A (real or complex) has $b-$resonant spectrum if the set $\sigma^+(A)$ is $b-$resonant.

With this in mind, in [19] it is proved the following result.

Theorem 8. *Let* $(A_n)_{n=1}^{\infty}$ *be* $p-$*periodic for some* $p\geq 1$ *and assume that the matrices* $A_1,\ldots,A_q$ *do not have* $b-$*resonant spectrums. Then for every* $c, x_0\in\mathbb{R}^d$*, the sequence* $c, \operatorname{Orb}(x_0)$ *is either finite or* $b-$*Benford.*

Open Question 7.

1. *Given the one dimensional dynamical system* (I,f)*, study the points in I whose trajectories satisfy the Benford's law. This means try to state the properties of these trajectories to have such property.*
2. *Consider the sequences composed by distances of pair of points and relate the above results with existence or not of distributional chaos (for definitions, see [14]).*
3. *Consider nonlinear non-autonomous systems and study their behaviour concerning the same property.*

3.2 On forbidden sets

In recent literature, there are an increasing number of papers where the forbidden sets of difference equations are computed. We review and complete different attempts to describe the forbidden set and propose new perspectives for further research and a list of open problems in this field.

The study of difference equations *(DE)* is an interesting and useful branch of discrete dynamical systems due to their variety of behaviours and their ability to model phenomena of applied sciences (see [24, 26, 36, 45] and references therein). The standard framework for this study is to consider iteration functions and sets of initial conditions in such a way that the values of the iterates belong to the domain of definition of the iteration function and therefore, the solutions are always well-defined. For example, in rational difference equations *(RDE)*, a common hypothesis is to consider positive coefficients and initial conditions, see [36, 45].
Such restrictions are also motivated by the use of *(DE)* as applied models, where negative initial conditions and/or parameters are usually meaningless [48].

But there is a recent interest to extend the known results to a new framework where initial conditions can be taken to be arbitrary real numbers and no restrictions are imposed to iteration functions. In this setting the forbidden set of a *(DE)* appears, namely, the set of initial conditions for which after a finite number of iterates we reach a value outside the domain of definition of the iteration function. Indeed, the central problem of the theory of *(DE)* is reformulated in the following terms:

Given a *(DE)*, determine the good $\mathfrak{G}$ and forbidden $\mathfrak{F}$ sets of initial conditions. For points in the good set, describe the dynamical properties of the solutions generated by them: boundedness, stability, periodicity, asymptotic behavior, etc.

Here, we are interested in the first part of the former problem: how to determine the forbidden set of a given *(DE)* of order k. In the previous literature to describe such sets, when it is achieved, it is usually interpreted as to be able to write a general term of a sequence of hypersurfaces in $\mathbb{R}^k$. But in those cases are precisely the corresponding to *(DE)* when it is also possible to give a general term for the solutions. Unfortunately, there are a little number of *(DEs)* with explicitly defined solutions. Hence, we claim that new qualitative perspectives must be assumed to deal with the problem above. Therefore, the goals in this subsection are the following: to organize several techniques used in the literature for the explicit determination of the forbidden set, revealing their resemblance in some cases, and giving some hints about how they can be generalized. Thus, we get a large list of *(DEs)* with known forbidden set that can be used as a frame to deal with the more ambitious problem to describe the forbidden set of a general *(DE)*. We review and introduce some methods to work also in that case. Finally, we propose some future directions of research.

The difference equation of Riccati plays a key role in this theory since as far as we know, almost all the literature where the forbidden set is described using a general term includes some kind of semiconjugacy involving such an equation. The *(DE)* obtained via a change of variables or topological semiconjugacy is a relevant tool. In the following, we will discuss how algebraic invariants can be used to transform a given equation into a Riccati or linear one depending upon a parameter, and therefore, determining its forbidden set.

After that, we will deal with an example of description, found in [25], where the elements of the forbidden set are given recurrently but explicitly.

It can be introduced a symbolic description of complex and real points of F, where we study some additional ways to deal with the forbidden set without an explicit formula.

Now we are concentrating in some problems stated in the recent literature concerning the above problems, in particular, in the non-autonomous Riccati difference equation of first order given by

$$x_{n+1} = \frac{a_n x_n + b_n}{c_n x_n + d_n}, \quad n = 0, 1, \ldots, \tag{$\star$}$$

where the sequences $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$, $(c_n)_{n=0}^{\infty}$, and $(d_n)_{n=0}^{\infty}$ are $q-$periodic and $x_0 \in \mathbb{R}$.

In [5], it is given a geometric description of the forbidden sets in terms of the coefficients of the equation in the general case and also in the following particular cases:

1. $b_n = 0$ and $d_n = 1$ for all $n = 0, 1, \ldots$ in both cases, when all the parameters are positive real numbers and they are general real numbers without restriction (see also [51]).

2. $a_n = 1$ for $n = 0, 1, \ldots$ and the sequence $(c_n)_{n=0}^{\infty}$ is not periodic.

Open Question 8.

1. *In ($\star$), describe the forbidden set in the cases when the sequences of coefficients are bounded and none of them is periodic.*

2. *The same in the cases when the maps $f_n = \frac{a_n x + b_n}{c_n x + d_n}$ are uniformly convergent to a map f, or alternatively, converge pointwise to f.*

3. *Solve the same problems that above through by another general nonlinear rational difference equations.*

4. *Face the same problems in the setting of systems of difference equations.*

4 Lyapunov exponents in non-autonomous systems

During years, a powerful tool to understand the behaviour and predictability in nonlinear discrete dynamical systems and time series obtained from them have been Lyapunov exponents. They have been used to decide when orbits are stable or instable in the Lyapunov sense. First, it is necessary to remark that while stability in the Lyapunov sense is a notion of topological character, Lyapunov exponents have a numerical nature and are calculated using the derivative of maps in the points of orbits.

It is an extended practice, particularly in experimental dynamics, to associate having trajectories with positive Lyapunov exponents with their instability and negative Lyapunov exponents with their stability. However, such interpretation has no firm mathematical foundation if some restrictions on the maps describing the dynamical systems are not introduced. In connection with such statement, in [31], they have been constructed two examples of autonomous dynamical systems defined by interval maps, one having a trajectory with positive Lyapunov exponent but stable and other having a trajectory with negative Lyapunov exponent but instable. But such maps are highly non-differentiable and therefore, we wonder if it is possible to obtain the same results via differentiable maps. In [11], they have been obtained such examples in the semi-open interval $[0, 1)$. We wonder if the above example can be constructed in the setting of non-autonomous systems.

In [8], it is introduced and applied for them an immediate extension of the definition of the Lyapunov exponents in the autonomous case (if the limit exists), when $X = I$. This definition is as follows.

Definition 7.

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \log |(f_{n-1} \circ \cdots \circ f_1 \circ f_0)'(x)| = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |f_j'(x_j)|,$$

and similarly for the strong Lyapunov exponent.

Stability and instability of trajectories are stated as the condition of being or not sensitive to initial conditions, namely,

Definition 8. The forward orbit $(x_n)_{n=0}^{\infty}$ is sensitive to initial conditions or Lyapunov instable if there exists $\varepsilon > 0$ such that for any $\delta > 0$, there exists y with $d(x_0, y) < \delta$ and $N \geq 0$ such that $d\left(f^N(y), f^N(x_0)\right) \geq \varepsilon$.

Definition 9. The forward orbit $(x_n)_{n=0}^{\infty}$ is not sensitive to initial conditions or it is Lyapunov stable if for any $\varepsilon > 0$ there is $\delta > 0$ such that whenever $d(x_0, y) < \delta$, then $d(f^n(y), f^n(x_0)) < \varepsilon$ for all $n \geq 0$.

One relevant situation is when $f_{0,\infty}$ is composed by a periodic sequence of maps of minimal period m, namely $f_{n+m} = f_n$ for every $n \geq 0$. Then the non-autonomous system is called *periodic of minimal period m* or simply *periodic of period m* (see [3, 27]).

To non-autonomous systems it is immediately extended the notion of instable trajectory or orbit in the Lyapunov sense.

Using the two maps f, g introduced in Section 4 of [11], we compute the Lyapunov exponent for the periodic case in the non-autonomous situation. Consider a periodic non-autonomous case of minimal period m composed by periodic blocks of repetitions of the former maps in any ordering. Let the map f be applied $0 < p < m$ times and $q = m - p$ times the map g. The case of *alternating maps* (see [3, 27]) holds when $p = q = 1$ and the block is $\{f, g, f, g, \ldots\}$.

Proposition 9. *For the periodic non-autonomous system in I given by repetition of the block composed by p times f and $m - p = q$ times g following any ordering, the trajectory of 0 has a strong Lyapunov exponent of value*

$$\Lambda(0) = \frac{p-q}{m} \log 2$$

and the orbit is instable.

Proof. Given any block of p times the map f and $p - q$ times g in any ordering and $p \neq q$, we use the former definitions to obtain the value of the Lyapunov exponent of the orbit of 0. The partial sums S_n of the corresponding series are given by

$$S_n = \frac{k(p-q) + i - j}{km + i + j} \log 2$$

where $0 < i \leq p$, $0 < j \leq q$, and $n = km + i + j$.
When $n \to \infty$, that series is convergent and it is immediate that its value is $\frac{p-q}{m} \log 2$, which is also its $\Lambda(0)$. Instability of the orbit is due to instability of the trajectory of $g(0)$.

Open Question 9.

1. *We claim that the orbit of initial condition $x_0 = 0$ would be instable if $q > 0$ and independently of this value. Moreover, we think that it is the case for non-periodic non-autonomous systems when the map g appears an infinite number of times, such as in the sequence $f_{0,\infty} = \{f, g, f, f, g, f, f, f, g, \ldots\}$.*

2. *Another way to choose the maps f and g is using the Thue-Morse sequence given by $\{0, 1, 1, 0, 1, 0, 0, 1, 0, \ldots\}$ (see [6] for more details). We choose the elements of the sequence. When we find one 0 we choose f and with 1, we choose g.*

3. *Solve the problem with Thue-Morse sequence.*

4. *Solve the problem for non-autonomous Hénon transformation (defined above).*

5. *Do the same in non-autonomous systems generated by perturbations of trigonometric and Jacobi types (see [8]).*

Acknowledgements

This research has been supported by Grant MTM2014-51891-P from Spanish MINECO.

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