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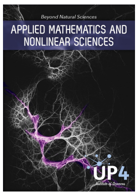


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# On $L^r$-regularity of global attractors generated by strong solutions of reaction-diffusion equations

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**Abstract**

In this paper we prove that the global attractor generated by strong solutions of a reaction-diffusion equation without uniqueness of the Cauchy problem is bounded in suitable $L^r$-spaces. In order to obtain this result we prove first that the concepts of weak and mild solutions are equivalent under an appropriate assumption.
Also, when the nonlinear term of the equation satisfies a supercritical growth condition the existence of a weak attractor is established.

**Keywords:** reaction-diffusion equation; setvalued dynamical system; global attractor; multivalued semiflow
**AMS 2010 codes:** 35B40, 35B41, 35K55, 35K57, 37B25, 58C06r

## 1 Introduction

This paper is mainly devoted to studying the regularity properties of global attractors for multivalued semiflows generated by strong solutions of reaction-diffusion equations.

The existence and properties of global attractors for dynamical systems generated by reaction-diffusion equations have been studied by many authors over the last thirty years. For equations generating a single-valued semigroup such results are well known since the 80s (see e.g. [5], [6], [7], [8], [24], [34]). Moreover, deep results concerning the structure of the attractors were proved for scalar equations (see [12], [13], [29], [30], [31]). Furthermore, for rather general parabolic equations boundedness of attractors in Sobolev spaces of higher orders were obtained as well (see e.g. [3], [4]).

When uniqueness of the Cauchy problem is not satisfied we have to work with multivalued semiflows rather than semigroups. In this direction, existence and topological properties of global and trajectory attractors have

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been studied by several authors over the last years (see [2], [9], [10], [11], [14], [16], [20], [22], [23], [25], [26], [36]). However, concerning the structure of the attractor little is known so far in comparison with the single-valued case. Nevertheless, recently the global attractor has been characterised using the unstable manifold of the set of stationary points, generalizing in this way well-known results from the single-valued case (see [17], [18], [19]). In particular, such structure was proved to be true for the global attractor generated by strong solutions of reaction diffusion equations in which the nonlinear term satisfies a critical growth condition.

It is important to point out that there are two different approaches to the study of these equations. One method relies on the construction of weak solutions through Galerkin approximations, whereas the other one makes use of the properties of sectorial operators in order to obtain mild solutions, which are defined by the variation of constants formula. It seems that this has given rise to two separate groups of papers, whose paths have rarely crossed. However, we find it interesting to use the powerful technique of sectorial operators in order to improve the regularity of weak solutions and global attractors in the multivalued case. Indeed, in this paper we prove that under a suitable assumption mild and weak solutions are equivalent, and using this result we are able to improve the regularity of the global attractor generated by strong solutions which was obtained in [17].

This paper is split into three different parts.

In the first section, we prove that the concepts of weak and mild solutions are equivalent provided that an appropriate condition holds.

In the second section, we use the above equivalence in order to show that the global attractor generated by strong solutions of the reaction-diffusion equation is bounded in suitable $L^r$-spaces in the case where the nonlinear term satisfies a critical growth condition.

Finally, in the third section, considering a supercritical growth condition we define a multivalued semiflow by taking all strong solutions satisfying an energy inequality and then prove that a weak global attractor exists, that is, we construct an attractor which attracts bounded sets of the phase space with respect to the weak topology.

## 2 Equivalence of different definitions of solutions

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be an open bounded subset with sufficiently smooth boundary $\partial\Omega$. We consider the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + f(t,u) = h(t,x), \quad x \in \Omega,\, t > \tau, \\ u|_{\partial\Omega} = 0, \\ u(\tau,x) = u_0(x),\, x \in \Omega. \end{cases} \tag{1}$$

The functions $f, h$ are assumed to satisfy the following conditions:

$$f \in \mathbb{C}(\mathbb{R} \times \mathbb{R}), \tag{2}$$

$$h \in L^2_{loc}(\mathbb{R}; L^2(\Omega)), \tag{3}$$

$$|f(t,u)| \leq C_1(1 + |u|^{p-1}), \tag{4}$$

$$f(t,u)u \geq \alpha |u|^p - C_2, \tag{5}$$

for all $(t,u) \in \mathbb{R} \times \mathbb{R}$, where $C_1, C_2$ are positive constants, $p \geq 2$ and $\alpha > 0$ if $p > 2$ but $\alpha \in \mathbb{R}$ if $p = 2$.

Denote $F(u) = \int_0^u f(s)ds$. If $p > 2$ from (4)-(5) we obtain that $\liminf_{|u|\to\infty} \frac{f(u)}{u} = +\infty$ and that there exists $D_1, D_2, \delta > 0$ such that

$$|F(u)| \leq D_1(1 + |u|^p),\ F(u) \geq \delta\, |u|^p - D_2, \quad \forall u \in \mathbb{R}. \tag{6}$$

If $p = 2$, then (6) remains valid but with $\delta \in \mathbb{R}$.

In the sequel we shall denote by $H$ the space $L^2(\Omega)$ endowed with the norm $\|\cdot\|$ and the scalar product $(\cdot,\cdot)$, and by $V$ the space $H_0^1(\Omega)$ with the norm $\|\cdot\|_V$ and the scalar product $((\cdot,\cdot))$, whereas $V' = H^{-1}(\Omega)$ is the dual space of $V$ with the norm $\|\cdot\|_{V'}$. The pairing between the space $V$ and $V'$ will be denoted by $\langle\cdot,\cdot\rangle$. Also, putting $\frac{1}{p}+\frac{1}{q}=1$, the pairing between the spaces $L^p(\Omega)$ and $L^q(\Omega)$ will be given by $\langle\cdot,\cdot\rangle_{q,p}$.

**Definition 1.** The function $u\in L^p_{loc}(\tau,+\infty;L^p(\Omega))\cap L^2_{loc}(\tau,+\infty;V)$ is called a weak solution to problem (1) on $(\tau,+\infty)$ if for all $T>\tau$, $v\in V\cap L^p(\Omega)$ one has

$$\frac{d}{dt}(u(t),v)+((u(t),v))+\langle f(t,u(t)),v\rangle_{q,p}=(h(t),v). \tag{7}$$

in the sense of scalar distributions on $(\tau,T)$.

For a weak solution let $g(t)=-f(t,u(t))+h(t)+\Delta u(t)$. From $u\in L^p_{loc}(\tau,+\infty;L^p(\Omega))\cap L^2_{loc}(\tau,+\infty;V)$ and condition (4) it is clear that $g\in L^q_{loc}(\tau,+\infty;L^q(\Omega))+L^2_{loc}(\tau,+\infty;V')$. Thus, we can rewrite (7) as

$$\frac{d}{dt}(u(t),v)=\langle g(t),v\rangle_{V'+L^q}, \text{ for all } v\in V\cap L^p(\Omega),$$

where $\langle\cdot,\cdot\rangle_{V'+L^q}$ denotes pairing between $V'+L^q(\Omega)$ and $V\cap L^p(\Omega)$.

It is well-known [10, p.284] that for any $u_\tau\in H$ there exists at least one weak solution $u(\cdot)$. If, moreover, $f(t,\cdot)\in C^1(\mathbb{R})$ and $\dfrac{\partial f}{\partial u}(t,u)\geq -C$ for any $(t,u)$, then the solution is unique.

In order to consider an equivalent equality to (7) we recall the following well-known result.

**Lemma 1.** *[33, p.250] Let $X$ be Banach space with dual $X'$, $u,g\in L^1(a,b;X)$. Then the following statements are equivalent:*

1. *$u(t)=\xi+\int_a^t g(s)ds$, $\xi\in X$ for a.a. $t\in(a,b)$;*
2. *$\int_a^b u(t)\varphi'(t)dt=-\int_a^b g(t)\varphi(t)dt$, $\forall\ \varphi\in C_0^\infty(a,b)$;*
3. *$\frac{d}{dt}\langle u,\eta\rangle_{X',X}=\langle g,\eta\rangle_{X',X}$, $\forall\ \eta\in X'$, in the sense of scalar distributions on $(a,b)$,*
   *where $\langle\cdot,\cdot\rangle_{X',X}$ denotes pairing between $X$ and $X'$.*

*If properties 1)-3) hold, then, moreover, $u(\cdot)$ is a.e. equivalent to a continuous function from $[a,b]$ in $X$.*

Applying this lemma with $X=V'+L^q(\Omega)$ we obtain that (7) is equivalent to the equality $\dfrac{du}{dt}=g$ in the sense of $X$-valued distributions on every interval $[\tau,T]$. Hence,

$$\frac{du}{dt}\in L^q_{loc}(\tau,+\infty;L^q(\Omega))+L^2_{loc}(\tau,+\infty;V') \tag{8}$$

and (7) is in fact equivalent to the equality

$$\int_\tau^T\left\langle\frac{du}{dt},\xi(t)\right\rangle_{V'+L^q}dt+\int_\tau^T((u(t),\xi(t)))\,dt+\int_\tau^T\langle f(t,u(t)),\xi(t)\rangle_{q,p}\,dt \tag{9}$$
$$=\int_\tau^T(h(t),\xi(t))\,dt,$$

for any $\xi\in L^p(\tau,T;L^p(\Omega))\cap L^2(\tau,T;V)$.

Property (8) implies that $u \in C([\tau,+\infty),H)$ and that the function $t \mapsto \|u(t)\|^2$ is absolutely continuous on $[\tau,T]$ and $\frac{d}{dt}\|u(t)\|^2 = 2\left\langle \frac{du}{dt},u\right\rangle_{V'+L^q}$ for a.a. $t \in (\tau,T)$ [10, p.285]. Hence, the initial condition $u(\tau) = u_\tau$ makes sense.

Also, since the space $C_0^\infty((\tau,T)\times\Omega)$ is dense in $L^p(\tau,T;L^p(\Omega))\cap L^2(\tau,T;V)$, equality (9) is equivalent to

$$\begin{aligned}
&-\int_\tau^T\int_\Omega u(t,x)\frac{\partial\phi}{\partial t}dxdt - \int_\tau^T\int_\Omega u(t,x)\frac{\partial^2\phi}{\partial x^2}dxdt + \int_\tau^T\int_\Omega f(u(t,x))\phi(t,x)dxdt \\
&= \int_\tau^T\int_\Omega h(t,x)\phi(t,x)dxdt,
\end{aligned}$$

for any $\phi \in C_0^\infty((\tau,T)\times\Omega)$, that is, $u(\cdot)$ is a weak solution if and only if it is a solution in the sense of distributions.

Let us consider now another concept of solution. Namely, we will define a mild solution to (1).

It is known that the operator $A = \Delta : D(A) = H^2(\Omega)\cap H_0^1(\Omega) \to H$ is the generator of a strongly continuous semigroup of contractions $T : \mathbb{R}^+ \times H \to H$. Moreover, it follows from well-known results [27] that for any $x \in D(A)$ the function $u(t) = T(t)x$ is the unique classical solution to the problem

$$\begin{cases} \dfrac{du}{dt} = Au(t),\ t > 0, \\ u(0) = x. \end{cases} \tag{10}$$

Further, we shall study the inhomogeneous problem

$$\begin{cases} \dfrac{du}{dt} = Au(t) + g(t),\ t > \tau, \\ u(\tau) = x, \end{cases} \tag{11}$$

where $g : \mathbb{R} \to H$.

**Definition 2.** Let $x \in H$ and $g \in L^1(\tau,T;H)$. Then the function $u \in C([\tau,T],H)$ is called a mild solution to problem (11) on $[\tau,T]$ if

$$u(t) = T(t-\tau)x + \int_\tau^t T(t-s)g(s)ds,\ \tau \le t \le T. \tag{12}$$

If $g \in L^1_{loc}([\tau,+\infty))$, then $u \in C([\tau,+\infty),H)$ is called a mild solution if (12) is satisfied on every interval $[\tau,T]$.

It is called a classical solution on $[\tau,T]$ if $u(\cdot)$ is continuously differentiable on $(\tau,T)$, $u(t) \in D(A)$ for any $t \in (\tau,T)$, $u(\tau) = x$ and the equality in (11) is satisfied for every $t \in (\tau,T)$.

It follows readily from this definition that problem (11) possesses a unique mild solution for every $x \in H$. Also, if $g$ is continuously differentiable on $[\tau,T]$ and $x \in D(A)$, then the mild solution is in fact the unique classical solution [27, p.107].

Coming back to our problem (1), let us introduce the concept of mild solution for it.

**Definition 3.** The function $u \in L^p_{loc}(\tau,+\infty;L^p(\Omega))\cap L^2_{loc}(\tau,+\infty;V)\cap C([\tau,+\infty),H)$ is called a mild solution to problem (1) on $(\tau,+\infty)$ if for all $T > \tau$ the function $g(\cdot) = h(\cdot) - f(\cdot,u(\cdot))$ belongs to $L^1(\tau,T;H)$ and the equality (12) holds true.

Our aim now is to show that under an additional assumption the concepts of weak and mild solutions coincide.

**Lemma 2.** *Assume that $u(\cdot)$ is a weak solution to problem (1) with initial datum $u_\tau \in H$ which satisfies*

$$f(\cdot, u(\cdot)) \in L^2(\tau, T; H). \tag{13}$$

*Then it is a mild solution as well.*

*Proof.* Let us define the function $g(t) = -f(t, u(t)) + h(t)$, which belongs to $L^2(\tau, T; H)$ for every $\tau < T$ due to (13). We need to prove that the equality (12) holds. For an arbitrary fixed interval $[\tau, T]$ we take sequences $u_0^n \in H^2(\Omega) \cap H_0^1(\Omega)$, $g^n(\cdot) \in C^1([\tau, \overline{T}], H)$, $T < \overline{T}$, such that $u_0^n \to u_0$ in $H$, $g^n \to g$ in $L^2(\tau, \overline{T}; H)$. Let $u^n$ be the unique classical solution to the problem

$$\begin{cases} \dfrac{du^n}{dt} = Au^n(t) + g^n(t), \ t \in (\tau, \overline{T}), \\ \qquad u^n(\tau) = u_\tau^n. \end{cases}$$

Since $u \in C^1([\tau + \varepsilon, T], H)$ for any $0 < \varepsilon < T - \tau$, the difference $u^n - u$ satisfies

$$\frac{1}{2}\frac{d}{dt}\|u^n - u\|^2 + \|u^n(t) - u(t)\|_V^2 \le \|g^n(t) - g(t)\| \, \|u^n(t) - u(t)\| \text{ for a.a. } t \in (\tau + \varepsilon, T).$$

Hence, by using Young's inequality and Gronwall's lemma it is standard to show that $u^n \to u$ in $C([\tau + \varepsilon, T], H)$.

Further, from the equality

$$u^n(t) = T(t - \tau - \varepsilon) u^n(\tau + \varepsilon) + \int_{\tau+\varepsilon}^t T(t - s) g^n(s) ds, \ \tau + \varepsilon \le t \le T,$$

and taking into account that

$$T(t - \tau - \varepsilon) u^n(\tau + \varepsilon) \to T(t - \tau - \varepsilon) u(\tau + \varepsilon),$$
$$\int_{\tau+\varepsilon}^t T(t - s) g^n(s) ds \to \int_{\tau+\varepsilon}^t T(t - s) g(s) ds,$$

where the last convergence follows from Lebesgue's theorem, we have

$$u(t) = T(t - \tau - \varepsilon) u(\tau + \varepsilon) + \int_{\tau+\varepsilon}^t T(t - s) g(s) ds, \ \tau + \varepsilon \le t \le T.$$

Passing to the limit as $\varepsilon \to 0$ and using that $T(t - \tau - \varepsilon) u(\tau + \varepsilon) \to T(t - \tau) u(\tau)$ we finally obtain that

$$u(t) = T(t - \tau) u(\tau) + \int_\tau^t T(t - s) g(s) ds, \ \tau \le t \le T,$$

so we conclude that $u(\cdot)$ is a mild solution.

We prove now the converse statement.

**Lemma 3.** *Assume that $u(\cdot)$ is a mild solution to problem (1) with initial data $u_\tau \in H$ which satisfies (13). Then it is a weak solution.*

*Proof.* Since the function $g(\cdot) = h(\cdot) - f(\cdot, u(\cdot))$ belongs to $L^2(\tau, T; H)$, using standard results [34, p.68] we obtain that the problem

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v = g(t, x), \quad x \in \Omega, \ t > \tau, \\ v|_{\partial\Omega} = 0, \\ v(\tau, x) = u_\tau(x), \ x \in \Omega, \end{cases}$$

possesses a unique weak solution $v(\cdot)$. However, arguing in the same way as in Lemma 2 we deduce that $v(\cdot)$ is a mild solution to problem (11) with $x = u_\tau$. Therefore, by uniqueness of the mild solution we have $u(\cdot) = v(\cdot)$, so our statement follows.

## 3 Regularity of the strong global attractor

Let us consider problem (1) in the autonomous case, that is,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + f(u) = h(x), \quad x \in \Omega,\ t > \tau, \\ u|_{\partial\Omega} = 0, \\ u(0,x) = u_0(x), x \in \Omega, \end{cases} \tag{14}$$

and we assume that conditions (2)-(5) are satisfied. In particular, this means that $h \in H$.

In [17] the existence of a global attractor for the multivalued semiflow generated by the strong solutions to (14) was proved assuming a critical growth condition on the nonlinear function $f$. In this section, our aim is to prove that this attractor is bounded in suitable $L^r$-spaces.

**Definition 4.** A weak solution $u(\cdot)$ to problem (1) is called a strong solution if, additionally, it satisfies

$$u \in L^\infty_{loc}(0,+\infty; V \cap L^p(\Omega)), \tag{15}$$

$$\frac{du}{dt} \in L^2_{loc}(0,+\infty; H). \tag{16}$$

As $u \in L^\infty(0,T;V \cap L^p(\Omega)) \cap C([0,T];H)$, we deduce that $u \in C([0,T];H^1_{0w}(\Omega) \cap L^p_w(\Omega))$ for any $T > 0$, where $H^1_{0w}(\Omega)$, $L^p_w(\Omega)$ are respectively the spaces $H^1_0(\Omega)$, $L^p(\Omega)$ with the weak topology [33, Lemma 1.4, p.263].

Throughout this section we will assume that

$$p \le \frac{2N-2}{N-2} \tag{17}$$

if $N \ge 3$, that is, $f$ satisfies a critical growth condition. No assumption is imposed if $N \le 2$.

From (4), (17) and the imbeddings $H^1(\Omega) \subset L^{\frac{2N}{N-2}}(\Omega)$, if $N \ge 3$, $H^1(\Omega) \subset L^q(\Omega)$, for any $q \ge 1$ if $N \le 2$, we get

$$\int_\tau^T \int_\Omega |f(u)|^2 dxdt \le K_1 \int_\tau^T \left(1 + \|u(t)\|^{2p-2}_{L^{2p-2}(\Omega)}\right) dt \le K_2 \int_\tau^T \left(1 + \|u(t)\|_V^{2p-2}\right) dt. \tag{18}$$

Then the equality $\Delta u = \dfrac{du}{dt} + f(u) - h$ and (15)-(16) imply that

$$u \in L^2(\tau, T; D(A)) \tag{19}$$

for any strong solution $u(\cdot)$.

Since $p \le \frac{2N-2}{N-2} \le \frac{2N}{N-2}$, it is also obvious that $V \subset L^p(\Omega)$.

By standard results [28, Corollary 7.3] we obtain then that $u \in C([0,+\infty), V)$ and

$$\frac{d}{dt}\|u\|_V^2 = 2\left(-\Delta u, \frac{du}{dt}\right) \text{ for a.a. } t > 0. \tag{20}$$

It is known [17] that for any $u_0 \in V$ there exists at least one strong solution $u(\cdot)$ such that $u(0) = u_0$. Moreover, every strong solution satisfies the energy inequality

$$E(u(t)) + 2\int_s^t \|u_r\|^2 dr = E(u(s)), \text{ for all } t \ge s \ge 0, \tag{21}$$

where $E(u(t)) = \|u(t)\|_V^2 + 2(F(u(t)),1) - 2(h, u(t))$.

Let

$$K_s^+ = \{u(\cdot) : u \text{ is a strong solution of (14)}\}.$$

We define now the multivalued map $G : \mathbb{R}^+ \times V \to P(V)$, where $P(V)$ is the set of all non-empty subsets of $V$, by

$$G(t,u_0) = \{u(t) : u \in K_s^+ \text{ and } u(0) = u_0\}.$$

We recall now some results proved in [17] about the existence and structure of a global attractor for $G$. It is worth pointing out that, although in that paper such theorems were proved in the particular three-dimensional case (i.e. $N = 3$), the proofs in the general $N$-dimensional setting are identical.

First, we note that $G$ is a strict multivalued semiflow, that is, $G(0,x) = x$ and $G(t+s,x) = G(t,G(s,x))$ for any $t,s \in \mathbb{R}^+$, $x \in V$.

Moreover, $G$ possesses a global compact invariant attractor $\mathscr{A}$, that is, $\mathscr{A}$ is compact in $V$, it is invariant (i.e. $\mathscr{A} = G(t,\mathscr{A})$ for any $t \geq 0$) and attracts every bounded set in $V$, that is,

$$dist(G(t,B),\mathscr{A}) \to 0 \text{ as } t \to +\infty,$$

for any $B$ set bounded in $V$, where $dist$ refers to the Hausdorff semidistance between sets given by $dist(A,B) = \sup_{x\in A} \inf_{y\in B} \|x-y\|_V$.

The map $\gamma : \mathbb{R} \to V$ is called a complete trajectory of $K_s^+$ if $\gamma(\cdot+h)\mid_{[0,+\infty)} \in K_s^+$ for any $h \in \mathbb{R}$, and this is equivalent to $\gamma$ being continuous and satisfying

$$\gamma(t+s) \in G_s(t,\gamma(s)) \text{ for all } s \in \mathbb{R} \text{ and } t \geq 0. \tag{22}$$

The set of all complete trajectories of $K_s^+$ will be denoted by $\mathbb{F}_s$. Let $\mathbb{K}_s$ be the set of all complete trajectories which are bounded in $V$.

The attractor $\mathscr{A}$ is characterised by the union of all points lying in a bounded complete trajectory, that is,

$$\mathscr{A} = \{\gamma(0) : \gamma(\cdot) \in \mathbb{K}_s\} = \cup_{t\in\mathbb{R}} \{\gamma(t) : \gamma(\cdot) \in \mathbb{K}_s\}. \tag{23}$$

A point $z \in X$ is a fixed point of $K_s^+$ if $\varphi(t) \equiv z \in K_s^+$, whereas it is called a fixed point of $G$ if $z \in G(t,z)$ for all $t \geq 0$. In our case these two concepts are equivalent, so we will simply call them fixed points. Moreover, $z$ is a fixed point if and only if $z \in V \cap H^2(\Omega)$ and

$$-\Delta z + f(z) = h \text{ in } H. \tag{24}$$

The set of all fixed points will be denoted by $\mathfrak{R}$.

Finally, in [17] it is proved that the strong global attractor coincides with the unstable manifold of the set of stationary points, and also with the stable one when we consider only bounded complete solutions. This means that

$$\Theta_s = M_s^+(\mathfrak{R}) = M_s^-(\mathfrak{R}), \tag{25}$$

where

$$\begin{aligned} M_s^-(\mathfrak{R}) &= \{z : \exists\gamma(\cdot) \in \mathbb{K}_s,\ \gamma(0) = z,\ \ dist(\gamma(t),\mathfrak{R}) \to 0,\ t \to +\infty\}, \\ M_s^+(\mathfrak{R}) &= \{z : \exists\gamma(\cdot) \in \mathbb{F}_s,\ \gamma(0) = z,\ \ dist(\gamma(t),\mathfrak{R}) \to 0,\ t \to -\infty\}. \end{aligned}$$

Concerning boundedness of the attractor in stronger $L^r$-spaces than $L^{\frac{2N}{N-2}}(\Omega)$ (which follows from the embedding $V \subset L^{\frac{2N}{N-2}}(\Omega)$) in [17] and [18] an estimate in the space $L^\infty(\Omega)$ is shown to be true if $h \in L^\infty(\Omega)$.

Now, using the equivalence between weak and mild solutions from the first section we are able to obtain suitable estimates in $L^r$-spaces under much weaker assumptions on the function $h$.

**Theorem 4.** *The global attractor $\mathscr{A}$ is bounded in $L^r(\Omega)$, where $r \in [1,+\infty]$ if $N \leq 3$, $r \in [1,+\infty)$ if $N = 4$ and $1 \leq r < \frac{2N}{N-4}$ if $N \geq 5$.*

*If $N \geq 4$ and $h \in L^{\overline{q}}(\Omega)$, for some $\overline{q} > \frac{N}{2}$, then $\mathscr{A}$ is bounded in $L^\infty(\Omega)$.*

*Proof.* The semigroup $T(t)$ generated by the solutions of problem (10) satisfies the following well-known estimate [32, p.2988]:

$$\|T(t)x\|_{L^r(\Omega)} \leq M\frac{e^{\delta t}}{t^{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}}\|x\|_{L^q(\Omega)}, \tag{26}$$

for every $1 \leq q \leq r \leq \infty$ and certain $M > 0$, $\delta \in \mathbb{R}$.

Take an arbitrary $y \in \mathscr{A}$. In view of (23), there exists a complete trajectory $\psi$ such that $\psi(0) = y$ and $\psi(t) \in \mathscr{A}$ for all $t \in \mathbb{R}$. Inequality (18), combined with (15) and Lemma 2, implies that $u(\cdot) = \psi(\cdot + h)$ is a mild solution to problem (14) for any $h \in \mathbb{R}$. We choose $h = -1$ and then by the variation of constants formula we have

$$y = u(1) = T(1)u(0) + \int_0^1 T(1-s)(h - f(u(s))ds.$$

Hence, applying (26) with $q = 2$ it follows that

$$\|y\|_{L^r(\Omega)} \leq Me^{\delta}\|u(0)\| + \int_0^1 M\frac{e^{\delta s}}{(1-s)^{\frac{N}{2}\left(\frac{1}{2}-\frac{1}{r}\right)}}(\|h\| + \|f(u(s)\|)ds$$

for any $r \geq 2$. Take an arbitrary value $r$ satisfying $r \in [2,+\infty]$ if $N \leq 3$, $r \in [2,+\infty)$ if $N = 4$ and $2 \leq r < \frac{2N}{N-4}$ if $N \geq 5$. The inequality $\|f(u(s)\|^2 \leq R_1(1 + \|u(t)\|_V^{2p-2})$, together with the boundedness of the attractor in the space $V$, gives the existence of a constant $R_2$ such that

$$\|y\|_{L^r(\Omega)} \leq R_2\left(1 + \int_0^1 \frac{1}{(1-s)^{\frac{N}{2}\left(\frac{1}{2}-\frac{1}{r}\right)}}ds\right) = R_2\left(1 + \frac{1}{\frac{N}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+1}\right),$$

where we have used that due to the conditions imposed on the parameter $r$, it follows that $\frac{N}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+1 > 0$.

Thus the first statement of the theorem is proved.

In order to prove the second one, we first will prove that if the global attractor is bounded in $L^s(\Omega)$, where $s$ satisfies

$$s\frac{N-2}{N} > \frac{N}{2}, \tag{27}$$

then it is bounded in fact in $L^\infty(\Omega)$. By (4) and (17) we get

$$\begin{aligned}\int_\Omega |f(u(s,x)|^{s\frac{N-2}{N}}dx &\leq R_3(1 + \int_\Omega |u(s,x)|^{s\frac{N-2}{N}(p-1)}dx) \\ &\leq R_3(1 + \int_\Omega |u(s,x)|^s dx) \leq R_4.\end{aligned} \tag{28}$$

Therefore, applying again formula (26) with $r = \infty$ and $q = \min\{\overline{q}, s\frac{N-2}{N}\}$ and arguing as before for any $y \in \mathscr{A}$ we have

$$\begin{aligned}\|y\|_{L^\infty(\Omega)} &\leq Me^{\delta}\|u(0)\|_{L^q(\Omega)} + \int_0^1 M\frac{e^{\delta s}}{(1-s)^{\frac{N}{2}\frac{1}{q}}}\left(\|h\|_{L^q(\Omega)} + \|f(u(s)\|_{L^q(\Omega)}\right)ds \\ &\leq R_5\left(1 + \int_0^1 \frac{1}{(1-s)^{\frac{N}{2}\frac{1}{q}}}ds\right) = R_5\left(1 + \frac{1}{-\frac{N}{2q}+1}\right),\end{aligned}$$

where we have used that $-\frac{N}{2q} + 1 > 0$. Thus, the result follows.

Observe that if $N = 4$, then the attractor is bounded in $L^s(\Omega)$ for an arbitrary $s \in [2,\infty)$. Hence, we can pick $s$ such that (27) holds. On the other hand, since we have proved that the attractor is bounded in $L^s(\Omega)$ for any $s < \frac{2N}{N-4}$, then (27) is also satisfied if $N = 5,6$.

Further, for $N \geq 7$ we will apply the above procedure iteratively so as to achieve (27).

Assume that the attractor is bounded in $L^s(\Omega)$ for some $s \geq \frac{2N}{N-2}$ such that $s\frac{N-2}{N} < \frac{N}{2}$. Using (28) and applying again (26) with $q = s\frac{N-2}{N}$ and $\bar{s} < \frac{s(N-2)N}{N^2-2s(N-2)}$ for any $y \in \mathscr{A}$ we obtain

$$\|y\|_{L^{\bar{s}}(\Omega)} \leq R_6 \left(1 + \int_0^1 \frac{1}{(1-s)^{\frac{N}{2}\left(\frac{N}{r(N-2)} - \frac{1}{\bar{s}}\right)}} ds\right) = \frac{R_6}{\frac{N}{2}\left(\frac{1}{\bar{s}} - \frac{N}{s(N-2)}\right) + 1}.$$

Therefore, the attractor is bounded in $L^{\bar{s}}(\Omega)$ as well. For an arbitrary $\varepsilon > 0$ we choose $\bar{s}$ such that the difference $\bar{s} - s$ satisfies

$$\bar{s} - s \geq \frac{s(N-2)N}{N^2 - 2s(N-2)} - s - \varepsilon.$$

Since $s \geq \frac{2N}{N-2}$, we get

$$\bar{s} - s \geq s\left(\frac{(N-2)N}{N^2 - 4N} - 1\right) - \varepsilon = s\frac{2}{N-4} - \varepsilon \geq \frac{4N}{(N-2)(N-4)} - \varepsilon.$$

There exist $\varepsilon > 0$ and $k \in \mathbb{N}$ such that

$$\frac{2N}{N-2} + (k-1)\left(\frac{4N}{(N-2)(N-4)} - \varepsilon\right) < \frac{N^2}{2(N-2)} < \frac{2N}{N-2} + k\left(\frac{4N}{(N-2)(N-4)} - \varepsilon\right)$$

and thus proceeding iteratively we obtain in $k$ steps that the global attractor is bounded in $L^s(\Omega)$, where $s$ satisfies (27).

## 4 Weak attractor for strong solutions in the supercritical case

In this section, our aim is to prove the existence of a weak global attractor for the multivalued semiflow generated by strong solutions to problem (14) satisfying a suitable energy inequality without imposing the assumption (17). In this case, we do not know whether strong solutions belong to the space of continuous functions with values in $V$, and therefore we are still not able to prove the existence of a strong attractor. Instead, we have to consider a weaker attractor in which the attracting property is satisfied with respect to the weak topology of the space $V \cap L^p(\Omega)$.

**Lemma 5.** *Let $u_\tau \in V \cap L^p(\Omega)$. Then there exists at least one strong solution $u$ of (1) such that $u(\tau) = u_\tau$. Moreover, the energy inequality*

$$E(u(t)) + \int_s^t \left\|\frac{du}{dr}\right\|^2 dr \leq E(u(s)) \tag{29}$$

*holds for all $t \geq s$, a.a. $s > 0$ and $s = 0$, where $E(u(t)) = \|u(t)\|_V^2 + (F(u(t)),1) - 2(h,u(t))$.*

*Proof.* As usual, we take the Galerkin approximations using the basis of eigenfunctions $\{w_j(x),\ j \in \mathbb{N}\}$ of $-\Delta$ with Dirichlet boundary conditions. Let $X_m = span\{w_1,...,w_m\}$ and let $P_m$ be the orthogonal projector from $H$ onto $X_m$. Then $u_m(t,x) = \sum_{j=i}^m a_{j,m}(t)\,w_j(x)$ will be a solution of the system of ordinary differential equations

$$\frac{du_m}{dt} = P_m\Delta u_m - P_m f(u_m) + P_m h,\ u_m(0) = P_m u_0. \tag{30}$$

It is proved in [10, p.281] that passing to a subsequence $u_m$ converges to a weak solution $u$ of (1) weakly star in $L^\infty(0,T;H)$, weakly in $L^p(0,T;L^p(\Omega))$ and weakly in $L^2(0,T;V)$ for all $T > 0$. Also, $u(\tau) = u_\tau$ and $\frac{du_m}{dt} \to \frac{du}{dt}$ weakly in $L^q(0,T;H^{-s}(\Omega))$ for some $s > 0$.

Multiplying (30) by $\frac{du_m}{dt}$ we get

$$\frac{d}{dt}\left(\|u_m\|_V^2+2(F(u_m),1)-2(h,u_m)\right)+2\|\frac{du_m}{dt}\|^2=0, \tag{31}$$

so (6) implies

$$\|u_m(t)\|_V^2+2\int_0^t\|\frac{d}{ds}u_m(s)\|^2ds+2\delta\|u(t)\|_{L^p(\Omega)}^p$$
$$\le\|u_m(0)\|_V^2+R_1\|u_m(0)\|_{L^p(\Omega)}^p+2\|h\|\,\|u_m(t)\|+2\|h\|\,\|u_m(0)\|+R_2.$$

If $p>2$, $\delta>0$ implies that

$$\|u_m(t)\|_V^2+\int_0^t\|\frac{d}{ds}u_m(s)\|^2ds+\|u(t)\|_{L^p(\Omega)}^p\le R_3\left(\|u_m(0)\|_V^2+\|u_m(0)\|_{L^p(\Omega)}^p\right)+R_4,$$

where $R_j>0$. When $p=2$ we obtain that

$$\|u_m(t)\|_V^2+\int_0^t\|\frac{d}{ds}u_m(s)\|^2ds\le R_3\|u_m(0)\|_V^2+R_4.$$

By the choise of the special basis we have that $u_m(0)\to u_0$ in $H_0^1(\Omega)\cap L^p(\Omega)$.

Hence, $u_m\to u$ weakly star in $L^\infty(0,T;V\cap L^p(\Omega))$ and $\frac{du_m}{dt}\to\frac{du}{dt}$ weakly in $L^2\left(0,T;L^2(\Omega)\right)$. Thus from the Ascoli-Arzelà theorem $\{u_m\}$ is pre-compact in $C([0,T];H)$ and then $u_m\to u$ in $C([0,T];H)$. Moreover, it follows that $u_m(t)\to u(t)$ in $H_{0w}^1(\Omega)\cap L_w^p(\Omega))$ for any $t\in[0,T]$.

Finally, we must check the validity of the energy inequality (29). It is clear from (31) that $u_m$ satisfy

$$E(u_m(t))+\int_s^t\left\|\frac{du_m}{dr}\right\|^2dr\le E(u_m(s))$$

for all $t\ge s\ge 0$.

Let us define the function

$$L(u)=f(u)u-\alpha|u|^p.$$

Multiplying the equation in (14) by $tu_m(t)$ and integrating over $(0,T)\times\Omega$ we obtain

$$\int_0^T\int_\Omega t\frac{du_m}{dt}u_m dxdt+\int_0^T t\|u_m(t)\|_V^2dt$$
$$+\int_0^T\int_\Omega tf(u_m(t,x))u_m(t,x)dxdt-\int_0^T\int_\Omega th(x)u_m(t,x)dxdt=0,$$

so

$$\begin{aligned}&\frac{T}{2}\|u_m(T)\|^2+\int_0^T t\|u_m(t)\|_V^2dt\\&+\int_0^T\int_\Omega tL(u_m(t,x))dxdt+\alpha\int_0^T t\|u_m(t)\|_{L^p(\Omega)}^p dt\\&=\frac{1}{2}\int_0^T\|u_m(t)\|^2dt+\int_0^T\int_\Omega th(x)u_m(t,x)dxdt.\end{aligned} \tag{32}$$

In the same way, for the limit function $u$ we obtain

$$\begin{aligned}&\frac{T}{2}\|u(T)\|^2+\int_0^T t\,\|u(t)\|_V^2\,dt\\&+\int_0^T\int_\Omega tL(u(t,x))dxdt+\alpha\int_0^T t\,\|u(t)\|_{L^p(\Omega)}^p\,dt\\&=\frac{1}{2}\int_0^T\|u(t)\|^2\,dt+\int_0^T\int_\Omega th(x)\,u(t,x)\,dxdt.\end{aligned}\tag{33}$$

From the previous convergences it is clear that

$$\int_0^T\|u_m(t)\|^2\,dt\to\int_0^T\|u(t)\|^2\,dt,$$
$$\int_0^T\int_\Omega th(x)\,u_m(t,x)\,dxdt\to\int_0^T\int_\Omega th(x)\,u(t,x)\,dxdt,$$

which implies that the left-hand side of (32) converges to the left-hand side of (33).

On the other hand, we have

$$\int_0^T\int_\Omega tL(u(t,x))dxdt\le\liminf\int_0^T\int_\Omega tL(u_m(t,x))dxdt,$$

which follows from the inequality

$$L(u_m(t,x))\ge -C_2,$$

the convergence $u_m(t,x)\to u(t,x)$, for a.a. $(t,x)$, and Lebesgue-Fatou's lemma [35].

Bearing also in mind that

$$\|u_m(T)\|^2\to\|u(T)\|^2,$$
$$\int_0^T t\,\|u(t)\|_V^2\,dt\le\liminf\int_0^T t\,\|u_m(t)\|_V^2\,dt,$$
$$\int_0^T t\,\|u(t)\|_{L^p(\Omega)}^p\,dts\le\liminf\int_0^T t\,\|u_m(t)\|_{L^p(\Omega)}^p\,dt,$$

we obtain readily that each term in the the left-hand side of (32) converges to the corresponding term in the left-hand side of (33).

Therefore,

$$\int_0^T t\,\|u_m(t)\|_{L^p(\Omega)}^p\,dt\to\int_0^T t\,\|u(t)\|_{L^p(\Omega)}^p\,dt,$$

$$\int_0^T t\,\|u_m(t)\|_V^2\,dt\to\int_0^T t\,\|u_m(t)\|_V^2\,dt,$$

and then for any $0<r<T$ we get

$$u_m\to u\text{ strongly in }L^2(r,T;V)\cap L^p(r,T;L^p(\Omega)),$$

so

$$u_m(t)\to u(t)\ \text{ in }V\cap L^p(\Omega)\text{ for a.a. }t\in(0,T).$$

Applying then the dominated convergence theorem we deduce that $F(u_m(t)) \to F(u(t))$ in $L^1(\Omega)$ for a.a. $t \in (0,T)$. On top of that, in view of $u_m \to u$ in $C([0,T],H)$, for any $t \in [0,T]$ it follows that $F(u_m(t,x)) \to F(u(t,x))$ for a.a. $x \in \Omega$. Hence, by the inequality

$$F(u_m(t,x)) \geq -D_2 \text{ if } p > 2,$$
$$F(u_m(t,x)) \geq \delta |u_m(t,x)|^2 - D_2 \text{ if } p = 2,$$

and Lebesgue-Fatou's lemma [35], we have

$$\int_\Omega F(u(t,x))dx \leq \liminf \int_\Omega F(u_m(t,x))dx.$$

We know also from $u_m(t) \to u(t)$ weakly in $V$ that

$$\|u(t)\|_V \leq \liminf \|u_m(t)\|_V \text{ for all } t \in [0,T].$$

Thus,

$$E(u(t)) \leq \liminf E(u_m(t)) \text{ for all } t \in [0,T].$$

Passing to the limit in (21) we obtain (29).

For any $u_0 \in V \cap L^p(\Omega)$ let $\mathscr{R}(u_0)$ be the set of strong solutions to problem (14) which satisfy the energy inequality (29) for all $t \geq s$ and a.a. $s > 0$. This set is non-empty for every initial datum $u_0 \in V \cap L^p(\Omega)$ due to Lemma 5. Denoting $X = V \cap L^p(\Omega)$ we define now the multivalued mapping $G : \mathbb{R}^+ \times X \to P(X)$ by

$$G(t,u_0) = \{y : y = u(t) \in \mathscr{R}(u_0)\}.$$

It is straightforward to check that $G(0,x) = x$ and $G(t+s,x) \subset G(t,G(s,x))$ for any $x \in X$ and $t,s \in \mathbb{R}^+$, that is, $G$ is a multivalued semiflow.

In the sequel, $X_w = H^1_{0w}(\Omega) \cap L^p_w(\Omega)$.

**Definition 5.** The set $\mathscr{A}$ is called a weak global attractor for the multivalued semiflow $G$ if the following properties hold:

1. $\mathscr{A}$ is bounded in $X$ and compact in $X_w$.
2. $\mathscr{A}$ is weakly attracting, that is, for any bounded set $B$ in $X$ and any neighborhood $\mathscr{O}$ of $\mathscr{A}$ in $X_w$ there exists $T(\mathscr{O})$ such that
$$G(t,B) \subset \mathscr{O} \text{ for all } t \geq T.$$
3. $\mathscr{A}$ is negatively semi-invariant, that is,
$$\mathscr{A} \subset G(t,\mathscr{A}) \text{ for any } t \geq 0.$$

It follows from this definition that if $K$ is a weakly closed set which is weakly attracting, then for any neighborhood $\mathscr{O}$ of $K$ in $X_w$ there exists $T(\mathscr{O})$ such that

$$\mathscr{A} \subset G(t,\mathscr{A}) \subset \mathscr{O} \text{ for all } t \geq T.$$

Hence, $\mathscr{A} \subset \mathscr{K}$ as $X_w$ is a Hausdorff topological space. This means that $\mathscr{A}$ is the minimal weakly closed set which is weakly attracting.

We recall that the set $B_0$ is said to be absorbing if for any bounded set $B$ in $X$ there exists $T(B)$ such that

$$G(t,B) \subset B_0 \text{ for any } t \geq T.$$

**Lemma 6.** *The multivalued semiflow $G$ possesses a bounded absorbing set $B_0 = \{v \in X : \|v\|_V^2 + \|v\|_{L^p(\Omega)}^p \leq R\}$, where $R$ is a positive constant.*

*Remark* 1. The set $B_0$ is weakly closed.

*Proof.* The function $E(u(t))$ satisfies (29) and multiplying the equation in (14) by $u$ we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|u\|_V^2 + \alpha \|u\|_{L^p(\Omega)}^p \leq C_2 + \frac{1}{2\lambda_1}\|h\|^2 + \frac{\lambda_1}{2}\|u\|^2.$$

Therefore,

$$\|u(t)\|^2 \leq e^{-\lambda_1 t}\|u(0)\|^2 + K_1, \text{ for all } t \geq 0,$$

where $K_1 = \frac{2C_2}{\lambda_1} + \frac{1}{\lambda_1^2}\|h\|^2$, and

$$\begin{aligned}\int_t^{t+r}\left(\|u\|_V^2 + \|u\|_{L^p(\Omega)}^p\right)dt &\leq K_2(\|u(t)\|^2 + r)\\ &\leq K_3(e^{-\lambda_1 t}\|u(0)\|^2 + 1 + r),\end{aligned}$$

for some positive constants $K_2, K_3$ and any $r > 0$, $t \geq 0$.

Integrating in (29) over $(t, t+r)$ we get

$$E(t+r) \leq \frac{K_3(e^{-\lambda_1 t}\|u(0)\|^2 + 1 + r)}{r}, \text{ for all } r > 0, t \geq 0.$$

Taking into account that

$$\begin{aligned}E(u(t)) &= \|u(t)\|_V^2 + (F(u(t)),1) - 2(h, u(t))\\ &\geq \|u(t)\|_V^2 + \delta\|u(t)\|_{L^p(\Omega)}^p - \|h\|^2 - \|u(t)\|^2 - D_2,\end{aligned}$$

it follows easily from the previous estimates the existence of a constant $R > 0$ such that the bounded set

$$B_0 = \{v \in X : \|v\|_V^2 + \|v\|_{L^p(\Omega)}^p \leq R\}$$

is absorbing for $G$.

The theory of global attractors in topological spaces for multivalued semiflows and processes has been developed for example in [1,15,21]. However, since the abstract conditions imposed in those papers are not met in our particular situation, we will prove the existence of the weak attractor from scratch.

**Theorem 7.** *The multivalued semiflow $G$ possesses a weak global attractor $\mathscr{A}$.*

*Proof.* Take an arbitrary bounded set $B$. We recall that the $\omega$-limit set of $B$ is given by

$$\omega(B) = \cap_{s \geq 0} cl_{X_w} \cup_{t \geq s} G(t, B).$$

Since the space $X_w$ satisfies the first axiom of countability, an equivalent definition for $\omega(B)$ is the following

$$\omega(B) = \left\{\begin{array}{c} y \in X : \text{ there exist sequences } y_n \in G(t_n, x_n),\, t_n \to +\infty,\, x_n \in X \\ \text{such that } y_n \to y \text{ in } X_w \end{array}\right\}.$$

Therefore, as $X$ is a reflexive Banach space and by Lemma 6 the set $\cup_{t \geq T(B)} G(t, B)$ is bounded for some $T(B)$, any sequence $y_n \in G(t_n, B)$ with $t_n \to +\infty$ has a weakly convergent subsequence. Thus, $\omega(B)$ is non-empty. It is obvious that $\omega(B)$ belongs to the absorbing set $B_0$, so it is bounded in $X$, and that it is weakly closed, so it is compact in $X_w$.

Further, let us check that $\omega(B)$ weakly attracts $B$. Assuming the opposite, there exists a neighborhood $\mathscr{O}$ of $\omega(B)$ and a sequence $y_n \in G(t_n, B)$, where $t_n \to +\infty$, such that $y_n \notin \mathscr{O}$. But this leads to a contradiction as from $\{y_n\}$ we can extract a converging subsequence whose limit belongs to $\omega(B)$.

It remains to prove that $\omega(B)$ is negatively semi-invariant.

First of all, let us consider a sequence of strong solutions $u^n(\cdot) \in \mathscr{R}(u_0^n)$ such that $u^n(t) \in B_0$ for any $t \geq 0$. In view of well-known results (see [20, Lemma 15] and [16, Theorem 3.11]), there exists a weak solution $u(\cdot)$ to problem (14) and a subsequence $u^{n_k}(\cdot)$ such that $u^{n_k} \to u$ strongly in $C([0,T], H)$, for all $T > 0$ (among other convergences). But $u^n(t)$ are uniformly bounded in $X$, so

$$u^{n_k} \to u \text{ weakly star in } L^\infty(0,T; V \cap L^p(\Omega)) \text{ for all } T > 0,$$

$$u^{n_k}(t) \to u(t) \text{ in } X_w \text{ for any } t \geq 0.$$

Also, inequalities (29), (6) and $E(u^n(s)) \leq C$, for any $s \geq 0$ and $n$, imply that $\frac{du^n}{dt}$ are bounded in $L^2(0,T;H)$ and then

$$\frac{du^{n_k}}{dt} \to \frac{du}{dt} \text{ weakly in } L^2(0,T;H).$$

Therefore, $u(\cdot)$ is in fact a strong solution to problem (14). On top of that, arguing in the same way as in Lemma 5 we obtain that

$$u^{n_k} \to u \text{ strongly in } L^2(r,T;V) \cap L^p(r,T;L^p(\Omega)), \text{ for any } 0 < r < T,$$

and also that (29) is satisfied for all $t \geq s$ and a.a. $s > 0$. Thereby, $u(\cdot) \in \mathscr{R}(u_0)$.

Consider now an arbitrary element $y \in \omega(B)$ and $t > 0$. Then there is a sequence $y_n \in G(t_n, x_n)$ with $t_n \to +\infty$, $x_n \in B$ such that $y_n \to y$ in $X_w$. In addition, we take $N(B)$ for which $G(s,B) \subset B_0$ if $s \geq t_n - t$ and $n \geq N$. Since $G(t_n, x_n) \subset G(t, G(t_n - t, x_n))$, there are $z_n \in G(t_n - t, x_n)$, $u^n(\cdot) \in \mathscr{R}(z_n)$ satisfying that $u^n(s) \in B_0$, for any $s \geq 0$, and $y_n = u^n(t)$. Passing to a subsequence $z_n$ converges in the space $X_w$ to some $z \in \omega(B)$. In light of the previous arguments there exists $u(\cdot) \in \mathscr{R}(z)$ such that $u(t) = y$. Thus, $y \in G(t,z) \subset G(t, \omega(B))$, which proves that $\omega(B)$ is negatively semi-invariant.

Finally, let $\mathscr{A} = \omega(B_0)$. Obviously, $\mathscr{A}$ is bounded in $X$, compact in $X_w$ and negatively semi-invariant. The weakly attracting property follows from the chain of inclusions

$$G(t,B) \subset G(t - T(B), G(T(B), B)) \subset G(t - T(B), B_0)$$

and the fact that $\mathscr{A}$ weakly attracts $B_0$.

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