Applied Mathematics and Nonlinear Sciences 1(1) (2016) 301–310

Applied Mathematics and Nonlinear Sciences

http://journals.up4sciences.org

Relative Equilibria in the 4-Vortex Problem Bifurcating from an Equilateral Triangle Configuration

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Submission Info

Communicated by Juan L.G. Guirao
Received 4th February 2016
Accepted 2nd June 2016
Available 2nd June 2016

Abstract

We study the relative equilibria in the 4-vortex problem when two of vorticities are equal to 1, and the other two equal to m are enough small. We prove that for $m > 0$ there is a unique concave kite relative equilibria. We also prove that there is a unique convex planar relative equilibria having two pairs of equal vorticities located at the adjacent vertices of the configuration and it is an isosceles trapezoid.

Keywords: n-vortex problem, relative equilibria, bifurcations of relative equilibria.
AMS 2010 codes: 76F20, 37C10.

1 Introduction

The equations of motion for the interaction of point vortices were introduced by Helmholtz [8] (1858). In that paper, Helmholtz was the first to make explicit key properties of those portions of a fluid in which the vorticity occurs. Although his research was motivated in part by his interest in the effect of friction within a fluid, he developed the theory restricted to the dynamics of a perfect incompressible fluid with vorticity. Towards the end of his article, he introduces the concept of parallel vertical infinitely thin vortex filaments, each of which contains an invariant amount of circulation. Equivalently, we can consider the trace of this family of filaments, the intersection of them with a plane perpendicular to all these points are known as point vortices, and we think that they play an analogue to the point masses in celestial mechanics.

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ISSN 2444-8656 doi:10.21042/AMNS.2016.1.00025

A good introduction to the problem and methods of vortex dynamics can be found in Aref (2007) [2] and Newton (2001) [12]. In this paper, we are particularly interested in the relative equilibria motions, which are the simplest solutions described in the problem of n–point vortices. In this case the dynamics of the vortices is uniform with respect to a fixed rotation center, or a uniform translation. The simplest relative equilibria are those where n–settings vortices form a regular polygon where all values at the vortices or vorticities are equal.

In 2001, Kossin and Schubert [10] performed numerical experiments that describe the evolution of thin annular rings with high vorticity as a model for the behavior observed in the eye of a hurricane. A polygonal rigid rotation has been found in the eye of hurricane research models and weather forecast in the Group at the National Hurricane Center for Atmospheric Research (see the website [4] to observe some simulations).

It is natural to explore configurations that rotate rigidly in a framework of dynamical systems, studying the planar relative equilibria of n-vortex problem. The case of three-vortices has been widely studied by Gröbli [5], Kossin and Schubert [10] and Hernández-Garduño and Lacomba [9]. Equilateral triangles are always relative equilibria. There is also collinear relative equilibria, which are determined by a cubic equation with coefficients which are linear in the vorticities. Depending on the parameters you can get one, two or three collinear relative equilibria. In 2008, R. Moeckel and M. Hampton [6] proved the finiteness of relative equilibria in the four-vortex problem. The case of four vortices with two pairs of equal vorticities was studied by Hampton, Roberts and Santoprete [7]. In 1988, Meyer and Schmidt [11] showed that there is a parametric family of relative equilibria relating to the problem of 4-vortices which becomes degenerate to a value of the parameter. This degeneration suggests that a bifurcation occurs.

The motivation of this work is to find new relative equilibria, when two of the vortices are fusing at the same point, we believe that this kind of results can be applied to the analysis of the hurricanes, particularly when they are born.

The paper is organized as follows: After the introduction, in Section 2 we give the background on this subject, stating the equations of motion and defining the different kinds of configurations for the relative equilibria that we are studying. Section 3 contains the main results of this paper, starting with an equilateral triangle configuration, where in the limit case $m = 0$, we fuse the massless particles and prove that for this limit case emanate a unique concave kite relative equilibria and a unique convex relative equilibria which is an isosceles trapezoid.

2 The n-vortex problem

The governing equations for a collection of n–vortices, each one with strength $\Gamma_i \in \mathbb{R}$ and located at $z_i = (x_i, y_i) \in \mathbb{R}^2$, are given by

$$\dot{z}_i = \sum_{i \neq j}^{n} \Gamma_j \frac{z_i - z_j}{\|z_i - z_j\|^2}. \tag{1}$$

A motion of n vortices is said to be a *relative equilibrium* if, and only if, there exists a real number $\lambda \neq 0$, called *angular velocity*, such that, for every i and for all time

$$\dot{z}_i = \lambda(z_i - c), \tag{2}$$

where $c = \dfrac{\sum_i^n \Gamma_i z_i}{\sum_i \Gamma_i}$ is the center of vorticity. Of course we are assuming that $\sum_i \Gamma_i \neq 0$, which is the case studied in this paper, when the sum vanishes it is a degenerate case, the interested reader can find its analysis for instance in [13].

It is easy to check that when the n–vortices have the same vorticity and they are located at the vertices of a regular n–gon, they form a relative equilibrium. We are interested in relative equilibria formed with two pairs of equals vorticities where the mass of one of them is sufficiently small.

We are interested in the study of non-collinear planar relative equilibria formed by 4 vortex located at the vertex of a regular polygon. In this paper we follow the work made by Corbera and Llibre for the case of central configurations in celestial mechanics [3]. In that article, they study all central configurations that can be obtained in the case $m_1 = m_2 > m_3 = m_4 = m > 0$ with m enough small. First, they analyzed the case when the bodies are not in collision and study the collinear and planar case with two or three masses equal to 0, and then they studied the central configurations in binary collision for the collinear case, equilateral triangle, and kite configuration. In this work we will point the similarities and differences with that paper.

Definition 1. Assume that $\mathbf{z} = (\mathbf{z_1}, \mathbf{z_2}, \mathbf{z_3}, \mathbf{z_4})$ is a relative equilibria of the planar 4 vortex problem:

- $\mathbf{z}$ is convex if none of the vorticities is located in the interior of the triangle formed by the others,
- $\mathbf{z}$ is concave if one of the bodies is in the interior of the triangle formed by the others,
- $\mathbf{z}$ is a kite relative equilibria if it has an axis of symmetry passing through two non-adjacent bodies,
- $\mathbf{z}$ is a rhombus if it is convex and the four exterior edges are equal to each other.

For the case of central configurations in celestial mechanics Albouy, Fu and Sun showed that when four particles (q_1, q_2, q_3, q_4) form a two-dimensional central configuration, which is a convex quadrilateral having $[q_1, q_2]$ and $[q_3, q_4]$ as diagonals [1]. This configuration is symmetric con respect to the axis $[q_3, q_4]$ if and only if $m_1 = m_2$. It is symmetric with respect to the axis $[q_1, q_2]$ if and only if $m_3 = m_4$. Using exactly the same kind of computations it is possible to get a similar result in the 4–vortex problem, we omit here the details.

The goal of this paper is look for new configurations of relative equilibria in the 4–vortex problem for which $\Gamma_1 = \Gamma_2$ and $\Gamma_3 = \Gamma_4$ is small enough. A point that we must clarify is about the singularity that appear when the motion start with one couple of masses in binary collision, we will do ahead in this paper.

3 Relative equilibria with vorticities $\Gamma_1 = \Gamma_2,\ \Gamma_3 = \Gamma_4 = m$ and m small

In this section we will give a description of relative equilibria for the planar 4-vortex problem where the vorticities are equal two a two and one couple of them have vorticities small enough.

Starting from an equilateral triangle configuration, where the vortices 3 and 4 are in binary collision and each vortex is located on the vertices of a triangle. After normalization, we can assume without loss of generality that the value of the vorticities are $\Gamma_1 = \Gamma_2 = 1$ and $\Gamma_3 = \Gamma_4 = m$ with m small enough.

Let be $z_1 = (-1, 0)$, $z_2 = (1, 0)$, $z_3 = (x_3, y_3)$ and $z_4 = (x_4, y_4)$ the positions of 4–point vortices with vorticities $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 respectively. We say that two planar relative equilibria are equivalent if one can be obtained by the other one by doing a rotation or interchanging the names of the vorticities Γ_3 y Γ_4. It is easy to verify that this is an equivalence relation, so from here on we look for classes of relative equilibria, that by short we will continue called them simply as relative equilibria.

Since the vortices 1 and 2 are fixed at z_1 and z_2, we define $\mathbf{s} = (x_3, y_3, x_4, y_4)$ as a relative equilibrium solution. A collision relative equilibrium solution is given by $\mathbf{sc} = (0, \sqrt{3}, 0, \sqrt{3})$.

The main result of this article is the following:

Theorem 1. *Let be $\Gamma_1 = \Gamma_2 = 1, \Gamma_3 = \Gamma_4 = m, z_1 = (-1, 0), z_2 = (1, 0),_3 = (x_3, y_3)$ and $z_4 = (x_4, y_4)$ the positions of the vortices $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 respectively. We assume that $x_3, y_3 \geq 0,\quad y_3 \geq y_4$. Then the relative equilibria for $m = 0$ given by $sc = (0, \sqrt{3}, 0, \sqrt{3})$ can be continued to*

1. *A unique family $(x_3(m), y_3(m), x_4(m), y_4(m))$ concave kite relative equilibria for $m > 0$ small where*

$$
\begin{aligned}
x_3(m) &= x_4(m) = 0,\\
y_3(\mu) &= \sqrt{3} + \frac{2}{3}m^{1/2} + \frac{68}{405}m^{3/2} + O(\mu^2),\\
y_4(\mu) &= \sqrt{3} - \frac{2}{3}m^{1/2} - \frac{68}{405}m^{3/2} + O(\mu^2).
\end{aligned}
$$

2. *A unique family $(x_3(m), y_3(m), x_4(m), y_4(m))$ of isosceles trapezoid relative equilibria for $m \in (0, 3/2)$ where*

$$\begin{aligned}
x_3(m) &= \frac{2}{\sqrt{3}}m^{1/2} - \frac{4}{3\sqrt{3}}m^{3/2} + O(m^2),\\
x_4(m) &= -x_3(m),\\
y_3(\mu) &= \sqrt{3} + O(m^2),\\
y_4(m) &= y_3(m).
\end{aligned}$$

Remark 1. The assumptions in the above Theorem are not of great significance, the other configurations that we can have can be obtained from the configurations given in the Theorem by doing either a rotation in dimension three or by interchanging the names of the vortices Γ_3 and Γ_4.

3.1 Equations for relative equilibria

The centre of vorticities for a relative equilibria with the hypothesis of Theorem 1 is

$$c = \left(\frac{m(x_3+x_4)}{2(m+1)}, \frac{m(y_3+y_4)}{2(m+1)}\right),$$

with this the equations (2) become

$$e_i = 0, \text{ for } i = 1, \dots, 8, \tag{3}$$

where

$$\begin{aligned}
e_1 &= -\frac{1}{2} - \frac{m(x_3+1)}{r_{13}^2} - \frac{m(x_4+1)}{r_{14}^2} + \lambda\left(1 + \frac{m(x_3+x_4)}{2(m+1)}\right),\\
e_2 &= \frac{1}{2} - \frac{m(x_3-1)}{r_{23}^2} - \frac{m(x_4-1)}{r_{24}^2} - \lambda\left(1 - \frac{m(x_3+x_4)}{2(m+1)}\right),\\
e_3 &= \frac{m(x_3+1)}{r_{13}^2} + \frac{m(x_3-1)}{r_{23}^2} + \frac{m(x_3-x_4)}{r_{34}^2} - \lambda\left(x_3 - \frac{m(x_3+x_4)}{2(m+1)}\right),\\
e_4 &= \frac{m(x_4+1)}{r_{14}^2} + \frac{m(x_4-1)}{r_{24}^2} + \frac{m(x_4-x_3)}{r_{34}^2} - \lambda\left(x_4 - \frac{m(x_3+x_4)}{2(m+1)}\right),\\
e_5 &= m\left(-\frac{y_3}{r_{13}^2} - \frac{y_4}{r_{14}^2} + \frac{\lambda(y_3+y_4)}{2(m+1)}\right),\\
e_6 &= m\left(-\frac{y_3}{r_{23}^2} - \frac{y_4}{r_{24}^2} + \frac{\lambda(y_3+y_4)}{2(m+1)}\right),\\
e_7 &= \frac{y_3}{r_{13}^2} + \frac{y_3}{r_{23}^2} + \frac{m(y_3-y_4)}{r_{34}^2} - \lambda\left(y_3 - \frac{m(y_3+y_4)}{2(m+1)}\right),\\
e_8 &= \frac{y_4}{r_{14}^2} + \frac{y_4}{r_{24}^2} + \frac{m(y_4-y_3)}{r_{34}^2} - \lambda\left(y_4 - \frac{m(y_3+y_4)}{2(m+1)}\right),
\end{aligned}$$

and

$$\begin{aligned}
r_{13} &= \sqrt{(x_3+1)^2 + y_3^2}, & r_{14} &= \sqrt{(x_4+1)^2 + y_4^2},\\
r_{23} &= \sqrt{(x_3-1)^2 + y_3^2}, & r_{24} &= \sqrt{(x_4-1)^2 + y_4^2},\\
r_{34} &= \sqrt{(x_3-x_4)^2 + (y_3-y_4)^2}.
\end{aligned}$$

Equations (3) are not defined in binary collision between the vortices. That is, when it has one of the following, $r_{13}=0, r_{14}=0, r_{23}=0, r_{24}=0$ o $r_{34}=0$.
The eight equations of (3) are not independent because

$$\begin{aligned} e_1+e_2+me_3+me_4&=0,\\ e_5+e_6+me_7+me_8&=0. \end{aligned} \tag{4}$$

We observe that equations (3) are not defined in binary collision among the vortices, that is, when one of the following relationships hold, $r_{13}=0, r_{14}=0, r_{23}=0, r_{24}=0$ or $r_{34}=0$.

Now we define

$$\begin{aligned} E_1&=e_1-e_2, & E_2&=e_3-e_2, & E_3&=e_4-e_2,\\ E_4&=e_5-e_6, & E_5&=e_6-e_7, & E_6&=e_8-e_6, \end{aligned}$$

system (3) taken inside the system (4) is equivalent to the system

$$E_i=0, \quad \text{para } i=1,\dots,6. \tag{5}$$

Solving for λ in equation $E_1=0$, and substituting its value in the other equations of (5) we obtain

$$F_i=0, \quad \text{para } i=1,\dots,5, \tag{6}$$

where

$$\begin{aligned}
F_1 &= -\frac{x_3}{2}+\frac{x_3+1}{r_{13}^2}+\frac{x_3-1}{r_{23}^2}+m\left(-\frac{x_3^2-1}{2r_{13}^2}+\frac{x_3^2-1}{2r_{23}^2}+\right.\\
&\qquad \left.\frac{x_3-x_4}{r_{34}^2}+\frac{(x_3+1)(x_4-1)}{2r_{24}^2}-\frac{(x_3-1)(x_4+1)}{2r_{14}^2}\right),\\
F_2 &= -\frac{x_4}{2}+\frac{x_4+1}{r_{14}^2}+\frac{x_4-1}{r_{24}^2}+m\left(-\frac{x_4^2-1}{2r_{14}^2}+\frac{x_4^2-1}{2r_{24}^2}+\right.\\
&\qquad \left.\frac{x_4-x_3}{r_{34}^2}-\frac{(x_3+1)(x_4-1)}{2r_{13}^2}-\frac{(x_3-1)(x_4+1)}{2r_{23}^2}\right),\\
F_3 &= m\left(-\frac{y_3}{r_{13}^2}+\frac{y_3}{r_{23}^2}+\frac{y_4}{r_{24}^2}-\frac{y_4}{r_{14}^2}\right),\\
F_4 &= \frac{y_3}{r_{13}^2}+\frac{y_3}{r_{23}^2}-\frac{y_3}{2}+m\left(-\frac{(x_3+1)y_3}{2r_{13}^2}+\frac{(x_3+1)y_3}{2r_{23}^2}-\frac{(x_4+1)y_3}{2r_{14}^2}+\right.\\
&\qquad \left.\frac{y_3-y_4}{r_{34}^2}+\frac{(x_4-1)y_3+2y_4}{2r_{24}^2}\right),\\
F_5 &= \frac{y_4}{r_{14}^2}+\frac{y_4}{r_{24}^2}-\frac{y_4}{2}+m\left(-\frac{(x_3+1)y_4}{2r_{13}^2}-\frac{(x_4+1)y_4}{2r_{14}^2}-\frac{(x_4+1)y_4}{2r_{24}^2}+\right.\\
&\qquad \left.\frac{y_4-y_3}{r_{34}^2}+\frac{(x_3-1)y_4+2y_3}{2r_{23}^2}\right),
\end{aligned}$$

3.2 Relative equilibria with $m = 0$

When $m = 0$ system (6) is equivalent to

$$\begin{aligned} G(x_3,y_3) &= 0, & G(x_4,y_4) &= 0, \\ H(x_3,y_3) &= 0, & G(x_4,y_4) &= 0, \end{aligned} \tag{7}$$

where

$$\begin{aligned} G(x,y) &= \frac{x-1}{(x-1)^2+y^2} + \frac{x+1}{(x+1)^2+y^2} - \frac{x}{2}, \\ H(x,y) &= \frac{y}{(x-1)^2+y^2} + \frac{y}{(x+1)^2+y^2} - \frac{y}{2}. \end{aligned}$$

Solving system (7) we find the solutions

$$(x,y) = \{(0,0),(0,\sqrt{3}),(0,-\sqrt{3}),(-\sqrt{5},0),(\sqrt{5},0)\}.$$

We will use only the solution $(x,y) = (0,\sqrt{3})$, which is in agreement with our assumptions that $x_3, y_3 \geq 0$, and that agrees with our original assumption that we have started with an equilateral triangle configuration with vortices 3 and 4 fusing in binary collision.

3.3 Relative equilibria for m small emanating from a relative equilibrium in collision for $m = 0$

System (6) is not well defined when $(x_3,y_3) = (x_4,y_4)$, so we have to transform it into a new system that is well defined and analytic in a neighbourhood of $(x_3,y_3) = (x_4,y_4)$. In this way, we first consider the following system of equations

$$\begin{aligned} G_1 &= F_1 + F_2 = 0, & G_2 &= F_4 + F_5 = 0, & G_3 &= F_3, \\ G_4 &= F_2 - F_1 = 0, & G_5 &= F_5 - F_4 = 0, \end{aligned} \tag{8}$$

which is equivalent to system (6). The first three equations of (8) are analytic with respect to all its variables in a neighbourhood of $(x_3,y_3) = (x_4,y_4)$ and $m = 0$. The last two equations of (8) are not analytic at these points because they contain the term

$$\frac{m}{r_{34}^2} = \frac{m}{(x_3-x_4)^2+(y_3-y_4)^2}.$$

This term is well defined when $m \to 0$ if $(x_3,y_3)-(x_4,y_4) = O(m^\beta)$ with $\beta \leq 1/2$. Let be $\mu = m^{1/2}$, then making the change of variable $(x_4,y_4) = (x_3,y_3) + \mu(X_4,Y_4)$, we obtain a new system of equations, which is analytic in a neighbourhood of the point $(x_3, y_3), \mu = 0$, and $(X_4, Y_4) \neq 0$. The new system is given by

$$\begin{aligned}
G_1 &= \frac{2(x_3-1)}{r_{23}^2} + \frac{2(x_2+1)}{r_{13}^2} - x_3 + \mathscr{O}(\mu), \\
G_2 &= \frac{2y_3}{r_{23}^2} + \frac{2y_3}{r_{13}^2} - y_3 + \mathscr{O}(\mu), \\
G_3 &= 2y_3 \left(\frac{1}{r_{23}^2} - \frac{1}{r_{13}^2} \right) \mu^2 + \mathscr{O}(\mu^{\ni}), \\
G_4 &= -\frac{X_4}{2} + \frac{X_4(-1+2x_3-x_3^2+y_3^2)+2(1+x_3)y_3Y_4}{r_{23}^4} \\
&\quad + \frac{X_4(-1+2x_3-x_3^2+y_3^2)-2(1+x_3)y_3Y_4}{r_{13}^4} + \frac{2X_4}{X_4^2+y_4^2}, \\
G_5 &= -\frac{Y_4}{2} + \frac{Y_4(-1+2x_3+x_3^2-y_3^2)-2(1+x_3)y_3X_4}{r_{13}^4} \\
&\quad + \frac{Y_4(-1+2x_3-x_3^2-y_3^2)+2(1-x_3)y_3X_4}{r_{23}^4} + \frac{2Y_4}{X_4^2+y_4^2}.
\end{aligned}$$

Now we consider the following system of equations

$$\begin{aligned}
&\overline{G}_1 = G_1 = 0, \qquad \overline{G}_2 = G_2 = 0, \qquad \overline{G}_3 = G_3/\mu^2 = 0, \\
&\overline{G}_4 = G_4/\mu = 0, \qquad \overline{G}_5 = G_5/\mu = 0,
\end{aligned} \tag{9}$$

which is analytic with respect to all variables in a neighbourhood of $(x_3, y_3) = (0,0)$ and $(X_4, Y_4) \neq 0$.

First, we calculate the solutions of (9) with $\mu = 0$. The solutions of the first three equations with $x_3, y_3 \geq 0$ are $(x_3, y_3) = (0,0), (0, \sqrt{3}, \sqrt{5}, 0)$. As we have remarked at the beginning of this section, we are interested in solutions where the vortices form an equilateral triangle with two of the vortices fusing in a binary collision. Hence, we substitute the solution $(x_3, y_3) = (0, \sqrt{3})$ into the last two equations of (3), so that the resulting system has the following four distinct real solutions:

$$\begin{aligned}
&\mathbf{sc}_1 = (0, \sqrt{3}, 0, 4/3), \qquad \mathbf{sc}_2 = (0, \sqrt{3}, 0, -4/3), \\
&\mathbf{sc}_3 = (0, \sqrt{3}, 4/\sqrt{3}, 0), \qquad \mathbf{sc}_4 = (0, \sqrt{3}, 4/\sqrt{3}, 0),
\end{aligned}$$

the components of $\mathbf{sc}_j$ are (x_3, y_3, X_4, Y_4).

In the next step we use the Implicit Function Theorem to continue the solution of system (3) with $\mu = 0$ to solutions with μ small and positive. Let be

$$D = \begin{vmatrix}
\frac{\partial \overline{G}_1}{\partial x_3} & \frac{\partial \overline{G}_1}{\partial y_3} & \frac{\partial \overline{G}_1}{\partial X_4} & \frac{\partial \overline{G}_1}{\partial Y_4} \\
\frac{\partial \overline{G}_2}{\partial x_3} & \frac{\partial \overline{G}_2}{\partial y_3} & \frac{\partial \overline{G}_2}{\partial X_4} & \frac{\partial \overline{G}_2}{\partial Y_4} \\
\frac{\partial \overline{G}_4}{\partial x_3} & \frac{\partial \overline{G}_4}{\partial y_3} & \frac{\partial \overline{G}_4}{\partial X_4} & \frac{\partial \overline{G}_4}{\partial Y_4} \\
\frac{\partial \overline{G}_5}{\partial x_3} & \frac{\partial \overline{G}_5}{\partial y_3} & \frac{\partial \overline{G}_5}{\partial X_4} & \frac{\partial \overline{G}_5}{\partial Y_4}
\end{vmatrix},$$

and $\mathbf{s} = (x_3, y_3, X_4, Y_4)$. Since the system of equations (9) is analytic with respect to all its variables in a neighbourhood of the points $\mathbf{sc}_j$ with $j = 1, 2, 3, 4$ and

$$D|_{\mu=0,\mathbf{s}=\mathbf{sc}_1} = -\frac{81}{64} = D|_{\mu=0,\mathbf{s}=\mathbf{sc}_2} \neq 0,$$
$$D|_{\mu=0,\mathbf{s}=\mathbf{sc}_3} = \frac{27}{64} = D|_{\mu=0,\mathbf{s}=\mathbf{sc}_4} \neq 0.$$

By the Implicit Function Theorem, in a small neighbourhood U of $\mu = 0$, we can find a unique analytic function $(x_3(\mu), y_3(\mu), X_4(\mu), Y_4(\mu))$, satisfying the following system o equations

$$\overline{G}_1 = 0,\ \overline{G}_2 = 0,\ \overline{G}_4 = 0,\ \overline{G}_5 = 0, \tag{10}$$

and such that $\mathbf{sc}_j = (x_3(0), y_3(0), X_4(0), Y_4(0))$ for all $j = 1, 2, 3, 4$.

A solution $\mathbf{s} = \mathbf{s}(\mu) = (x_3(\mu), y_3(\mu), X_4(\mu), Y_4(\mu))$ of equations (10) expressed as a Taylor series takes the form

$$x_3(\mu) = \sum_{k=0}^{\infty} x_{3k}\mu^k, \qquad X_4(\mu) = \sum_{k=0}^{\infty} X_{4k}\mu^k,$$
$$y_3(\mu) = \sum_{k=0}^{\infty} y_{3k}\mu^k, \qquad Y_4(\mu) = \sum_{k=0}^{\infty} Y_{4k}\mu^k.$$

Expanding the equations $\overline{G}_1, \overline{G}_2, \overline{G}_4$ and $\overline{G}_5$ as a Taylor series around the point $\mathbf{s} = \mathbf{s}(\mu)$ with $(x_{30}, y_{30}, X_{40}, Y_{40}) = \mathbf{sc}_j$, and truncating the solutions at order four in μ, yields the following results:

If $\mathbf{sc}_1 = (0, \sqrt{3}, 0, 4/3)$ then

$$x_3(\mu) = 0 + O(\mu^4), \qquad X_4(\mu) = 0 + O(\mu^4),$$
$$y_3(\mu) = \sqrt{3} - \frac{2}{3}\mu + \frac{68}{405}\mu^3 + O(\mu^4), \qquad Y_4(\mu) = \frac{4}{3} - \frac{136}{405}\mu^2 + O(\mu^4).$$

These values do not provide solutions of (6) with $y_3 \geq y_4$ and so we discard them.

If $\mathbf{sc}_1 = (0, \sqrt{3}, 0, -4/3)$ then

$$x_3(\mu) = 0 + O(\mu^4), \qquad X_4(\mu) = 0 + O(\mu^4),$$
$$y_3(\mu) = \sqrt{3} + \frac{2}{3}\mu - \frac{68}{405}\mu^3 + O(\mu^4), \qquad Y_4(\mu) = -\frac{4}{3} + \frac{136}{405}\mu^2 + O(\mu^4).$$

After the following change of variables $(x_4, y_4) = (x_3, y_3) + \mu(X_4, Y_4)$ we obtain

$$x_3(\mu) = 0 + O(\mu^4), \qquad x_4(\mu) = 0 + O(\mu^4),$$
$$y_3(\mu) = \sqrt{3} + \frac{2}{3}\mu + \frac{68}{405}\mu^3 + O(\mu^4), \qquad y_4(\mu) = \sqrt{3} - \frac{2}{3}\mu - \frac{68}{405}\mu^3 + O(\mu^4),$$

so $x_3 = x_4 = 0$ and $y_3 \geq y_4$, therefore, the new family is a concave kite (see Fig. 1).

If $\mathbf{sc}_1 = (0, \sqrt{3}, 4/\sqrt{3}, 0)$ we have

$$x_3(\mu) = \frac{-2}{\sqrt{3}}\mu + \frac{4}{3\sqrt{3}}\mu^3 + O(\mu^4), \qquad X_4(\mu) = \frac{4}{\sqrt{3}} - \frac{8}{3\sqrt{3}}\mu^2 + O(\mu^4),$$
$$y_3(\mu) = \sqrt{3} + O(\mu^4), \qquad Y_4(\mu) = 0 + O(\mu^4).$$

We discard this solution since it fails to satisfy the assumption that $x_3 \geq 0$.

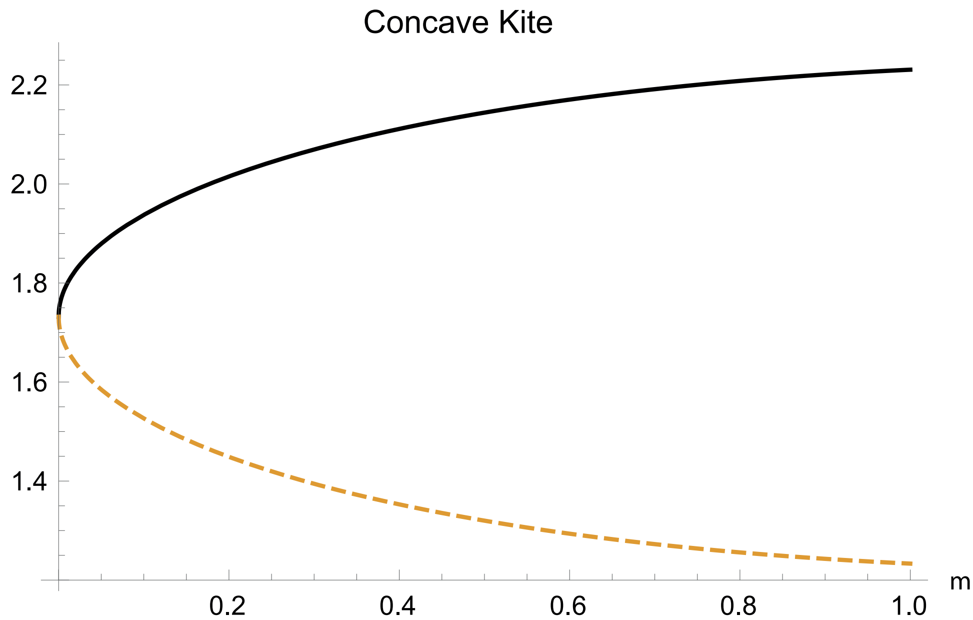

Fig. 1 Graph of functions $y_3(m)$ (continuos line) and $y_4(m)$ (dashed line), for $m > 0$.

If $\mathbf{sc}_1 = (0, \sqrt{3}, -4/\sqrt{3}, 0)$ then

$$x_3(\mu) = \frac{2}{\sqrt{3}}\mu - \frac{4}{3\sqrt{3}}\mu^3 + O(\mu^4), \qquad X_4(\mu) = -\frac{4}{\sqrt{3}} + \frac{8}{3\sqrt{3}}\mu^2 + O(\mu^4),$$
$$y_3(\mu) = \sqrt{3} + O(\mu^4), \qquad Y_4(\mu) = 0 + O(\mu^4).$$

Making the change of variables $(x_4, y_4) = (x_3, y_3) + \mu(X_4, Y_4)$ yields

$$x_3(\mu) = \frac{2}{\sqrt{3}}\mu - \frac{4}{3\sqrt{3}}\mu^3 + O(\mu^4), \qquad x_4(\mu) = -\frac{2}{\sqrt{3}}\mu + \frac{4}{3\sqrt{3}}\mu^3 + O(\mu^4),$$
$$y_3(\mu) = \sqrt{3} + O(\mu^4), \qquad y_4(\mu) = \sqrt{3} + O(\mu^4).$$

We can observe that $x_3 = -x_4$, $y_3 = y_4$ and $x_3 \geq 0$ if $m \in (0, 3/2)$, so the new family that emanates from the relative equilibrium in collision is an isosceles trapezoid (see Fig. 2).

Summarizing, all non-collinear relative equilibria in binary collision with the small mass $m = 0$ having an equilateral triangle shape can be continued to a concave kite configuration or to an isosceles trapezoid. There are not more possibilities. With all the above, we have proved Theorem 1. □

Acknowledgements

The first author (EPC) has been partially supported by the Asociación Mexicana de Cultura A.C. This paper is part of Claudia Tamayo Ph.D. thesis in the Program: "Doctorado en Ciencias (Matemáticas)" at UAM-Iztapalapa, she thanks the support given by Conacyt-México.

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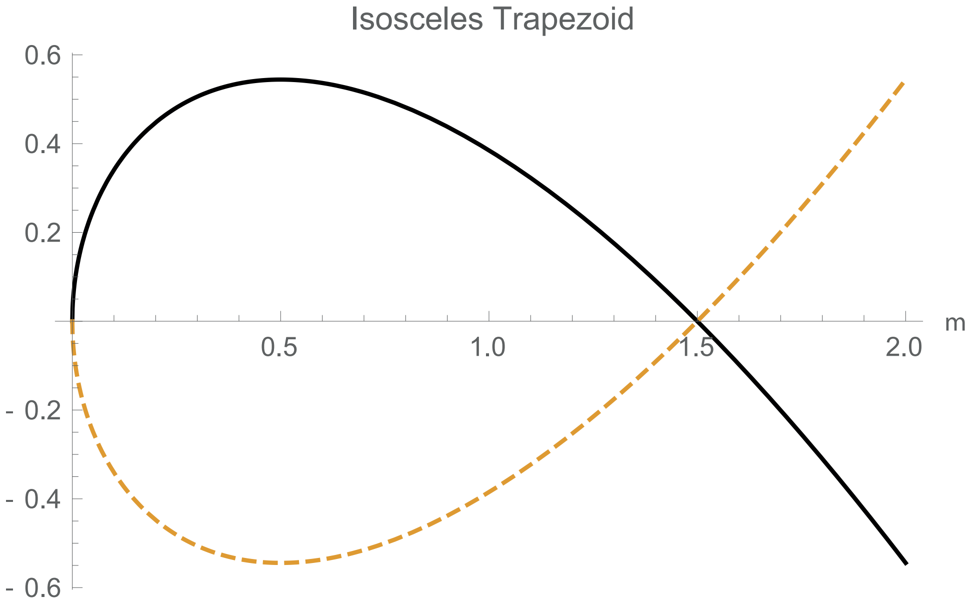

Fig. 2 Graph of functions $x_3(m)$ (continuous line) and $x_4(m)$ (dashed line).

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