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On the classical and nonclassical symmetries of a generalized Gardner equation

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Abstract

In this paper, we consider a generalized Gardner equation from the point of view of classical and nonclassical symmetries in partial differential equations. We perform a complete analysis of the symmetry reductions by using the similarity variables and the similarity solutions which allow us to reduce our equation into an ordinary differential equation. Moreover, we prove that the nonclassical method applied to the equation leads to new symmetries, which cannot be obtained by using the Lie classical method. Finally, we calculate exact travelling wave solutions of the equation by using the simplest equation method.

Keywords: Partial differential equations, Symmetries, Exact solutions
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1 Introduction

Many nonlinear phenomena are described by differential equations, especially by partial differential equations (PDEs). Symmetry group analysis of a differential equation appears as a powerful method to analyse PDEs [11] and fractional PDEs [13, 17]. Among its many applications, we highlight the fact that they allow us to obtain exact solutions of a PDE, directly or by using similarity solutions [1, 14]; classify invariant equations; reduce the number of independent variables and construct conservation laws [4, 7, 12, 21].

The symmetry group of a PDE is defined as the largest group of transformations acting on independent and dependent variables which transforms solutions of the equation into other solutions. Probably the most famous method used to obtain symmetries is the Lie classical method. The basic idea of Lie method is that, when a

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differential equation is invariant under a Lie group of transformations, there is a transformation which reduces the number of independent variables, in the case of a PDE; either the order of the equation, in the case of an ordinary differential equation (ODE).

Nevertheless, symmetry reductions for many PDEs are unobtainable by using classical symmetries. Accordingly, several generalizations of the Lie classical method have been established, for instance, the nonclassical method of Bluman and Cole [2]. This method generalizes the Lie classical method, however it is much more difficult to implement due to it leads to a system of nonlinear determining equations. This, couple with the fact that for some equations, such as the Korteweg-de Vries, the nonclassical method does not lead to new symmetries, it is appropriate to apply previously the Lie classical method. Many authors have used the nonclassical method to solve PDEs. In [10] Clarkson and Mansfield proposed an algorithm for calculating the determining equations associated to the nonclassical method. A different procedure for finding nonclassical symmetries was proposed by Bilă and Niesen in [3]. In [4] Bruzón and Gandarias extended the algorithm described for Bilă and Niesen to determine the nonclassical symmetries of a PDE for the case $\xi_p = 0$.

In the last decades travelling wave solutions of nonlinear PDEs have been studied [5]. A very successful method to obtain exact travelling wave solutions of numerous nonlinear PDEs is the method of simplest equation, especially its version called modified method of simplest equation. Simplest equation method is based on a procedure analogous to the first step of the test for the Painlevè property. In the modified simplest equation method, this procedure is substituted by the concept of balance equations [15, 16].

In this paper, we consider a generalized Gardner equation given by

$$u_t + a\,u^n\,u_x + b\,u^{2n}\,u_x + c\,u_{xxx} + d\,u_x + e\,u + f = 0, \tag{1}$$

where n is a positive constant, a and b are not simultaneously equal to zero, $c \neq 0$, d, e and f are arbitrary constants.

The Gardner equation, also known as combined KdV-mKdV equation, is widely used in various areas of physics, such that plasma physics, fluid dynamics, quantum field theory, and it is a useful model for the description of a great variety of wave phenomena in plasma and solid state. The Gardner equation has been recently considered by different authors [6, 12, 18, 20].

The aim of this paper is to study equation (1) from the point of view of symmetry reductions in PDEs. We obtain the classical and nonclassical symmetries of equation (1). Using the optimal system, we get the similarity variables and the similarity solutions which allow us to transform our equation into an ordinary differential equation. From these reductions we derive exact travelling waves solutions by using the simplest equation method given by Kudryashov. Some concluding remarks will end the paper.

2 Classical symmetries

In order to obtain the classical symmetries of equation (1) we apply the Lie classical method. This method is based on the determination of the symmetry group of a differential equation. For equation (1), a general element of the symmetry group is given by

$$\mathbf{v} = \tau(x,t,u)\partial_t + \xi(x,t,u)\partial_x + \eta(x,t,u)\partial_u. \tag{2}$$

The invariance of equation (1) under the infinitesimal generator (2) leads us to a set of determining equations for the unknown infinitesimals $\tau(x,t,u)$, $\xi(x,t,u)$ and $\eta(x,t,u)$ [19]. Simplifying this system we obtain that

$\tau=\tau(t)$, $\xi=\xi(x,t)$ and $\eta=\eta(x,t,u)$ must satisfy the following conditions:

$$\begin{aligned} \eta_{uu}&=0,\\ \eta_{ux}-\xi_{xx}&=0,\\ \tau_t-3\xi_x&=0,\\ 2b\,\xi_x u^{2n+1}+2bn\,\eta\, u^{2n}+2a\,\xi_x u^{n+1}+an\,\eta\, u^n+3c\,\eta_{uxx}u-c\,\xi_{xxx}u+2d\,\xi_x u-\xi_t u&=0,\\ b\,\eta_x u^{2n}+a\,\eta_x u^n-e\,\eta_u u+3e\,\xi_x u+c\,\eta_{xxx}+d\,\eta_x-f\,\eta_u+\eta_t+e\,\eta+3\,\xi_x f&=0. \end{aligned} \tag{3}$$

From determining system (3), if a, b, c, d, e, f and n are arbitrary, we get

$$\mathbf{v}_1=\partial_x, \qquad \mathbf{v}_2=\partial_t. \tag{4}$$

In the following cases, we obtain additional symmetries.

Case 1: $e\neq 0$

1.1. If $n=\frac{1}{2}$, $a=0$, $b=k$ or $n=1$, $a=k$, $b=0$,

$$\mathbf{v}_3=-\frac{k}{e}\exp(-et)\partial_x+\exp(-et)\partial_u. \tag{5}$$

Case 2: $e=0$

2.1. If $n=\frac{1}{2}$, $a=0$, $b=k$ or $n=1$, $a=k$, $b=0$,

$$\mathbf{v}_4=\left(x+2dt-\frac{5}{2}kft^2\right)\partial_x+3t\partial_t-(5ft+2u)\,\partial_u, \tag{6}$$

$$\mathbf{v}_5=kt\partial_x+\partial_u. \tag{7}$$

2.2. If $n\neq\frac{1}{2},1$, $a=0$, $f=0$,

$$\mathbf{v}_6=(x+2dt)\,\partial_x+3t\partial_t-\frac{u}{n}\partial_u. \tag{8}$$

2.3. If $n\neq 1$, $b=0$, $f=0$,

$$\mathbf{v}_7=(x+2dt)\,\partial_x+3t\partial_t-\frac{2u}{n}\partial_u. \tag{9}$$

2.4. If $n=1$, $f=0$,

$$\mathbf{v}_8=\left(2bx+4bdt-a^2t\right)\partial_x+6bt\partial_t-(2bu+a)\,\partial_u. \tag{10}$$

In the previous cases, $k\neq 0$ represents an arbitrary constant.

Suppose that $\mathscr{A}$ is an r-dimensional Lie algebra, and $\mathbf{v}_1,\mathbf{v}_2,\ldots\mathbf{v}_r$, form a basis for $\mathscr{A}$. If we consider that two subalgebras are related by a transformation of the group of symmetries, invariant solutions calculated from them will be related by the same transformation. We construct the optimal system of subalgebras to obtain those invariant solutions that cannot be derived from others. By using the optimal system, we calculate the similarity variables and the similarity solutions. This allows us to transform equation (1) into an ODE, solving the characteristic system

$$\frac{dx}{\xi}=\frac{dt}{\tau}=\frac{du}{\eta}. \tag{11}$$

In Table 1, we show the elements of the optimal system for each case in the symmetry classification along with their corresponding similarity variables and similarity solutions. Furthermore, in Table 2 we present the corresponding reduced ODEs.

Table 1 Similarity solutions and similarity variables of equation (1)

Subcase	Optimal system of subalgebras	Similarity variables	Similarity solutions
arbitrary	$<\lambda\mathbf{v}_1+\mu\mathbf{v}_2>$	$z=\mu x-\lambda t$	$u=h(z)$
1.1. a)	$<\lambda\mathbf{v}_1+\mu\mathbf{v}_2>$	$z=\mu x-\lambda t$	$u=h(z)$
1.1. b)	$<\lambda\mathbf{v}_1+\mu\mathbf{v}_3>$	$z=t$	$u=h(z)+\dfrac{\mu\exp(-et)x}{\lambda-\frac{k\mu}{e}\exp(-et)}$
2.1. a)	$<\lambda\mathbf{v}_1+\mu\mathbf{v}_2>$	$z=\mu x-\lambda t$	$u=h(z)$
2.1. b)	$<\lambda\mathbf{v}_1+\mu\mathbf{v}_5>$	$z=t$	$u=\dfrac{h(z)+\mu x}{\lambda+\mu kt}$
2.1. c)	$<\mathbf{v}_4>$	$z=(x-dt)t^{-\frac{1}{3}}+\dfrac{fkt^{\frac{5}{3}}}{2}$	$u=t^{-\frac{2}{3}}h(z)-ft$
2.2. a)	$<\lambda\mathbf{v}_1+\mu\mathbf{v}_2>$	$z=\mu x-\lambda t$	$u=h(z)$
2.2. b)	$<\mathbf{v}_6>$	$z=(x-dt)t^{-\frac{1}{3}}$	$u=t^{-\frac{1}{3n}}h(z)$
2.3. a)	$<\lambda\mathbf{v}_1+\mu\mathbf{v}_2>$	$z=\mu x-\lambda t$	$u=h(z)$
2.3. b)	$<\mathbf{v}_7>$	$z=(x-dt)t^{-\frac{1}{3}}$	$u=t^{-\frac{2}{3n}}h(z)$
2.4. a)	$<\lambda\mathbf{v}_1+\mu\mathbf{v}_2>$	$z=\mu x-\lambda t$	$u=h(z)$
2.4. b)	$<\mathbf{v}_8>$	$z=xt^{-\frac{1}{3}}+\dfrac{(a^2-4bd)t^{\frac{2}{3}}}{4b}$	$u=t^{-\frac{1}{3}}h(z)-\dfrac{a}{2b}$

3 Nonclassical symmetries

We apply the nonclassical method of Bluman and Cole [2] to get nonclassical symmetries of equation (1). Let us consider a s-th order PDE with p independent variables, $x=(x_1,\dots,x_p)$, and one dependent variable, $u=u(x)$,

$$\Delta\equiv\Delta(x,u,\mathbf{u}^{(1)}(x),\dots,\mathbf{u}^{(s)}(x))=0, \tag{12}$$

Table 2 Reduced equations

Subcase	ODEs
arbitrary	$c\mu^3 h''' + b\mu h^{2n} h' + a\mu h^n h' + (d\mu - \lambda)h' + eh + f = 0$
1.1. a)	$c\mu^3 h''' + k\mu hh' + (d\mu - \lambda)h' + eh + f = 0$
1.1. b)	$\lambda\, e\, \exp(e\,z)\,(h' + f) - \mu\,\left(e^2\,x + k\,h' - e\,h\,k + f\,k - d\,e\right) = 0$
2.1. a)	$c\mu^3 h''' + k\mu hh' + (d\mu - \lambda)h' + f = 0$
2.1. b)	$h' + \mu\, d + (\lambda + \mu\, k\, z)\, f = 0$
2.1. c)	$3\,c\,h''' + 3\,k\,h\,h' - h'\,z - 2\,h = 0$
2.2. a)	$c\mu^3 h''' + b\mu h^{2n} h' + (d\mu - \lambda)h' = 0$
2.2. b)	$3\,c\,n\,h''' + 3\,k\,n\,h^{2n}\,h' - n\,h'\,z - h = 0$
2.3. a)	$c\mu^3 h''' + \mu a h^n h' + (d\mu - \lambda)h' = 0$
2.3. b)	$3\,c\,n\,h''' + 3\,a\,n\,h^n\,h' - n\,h'\,z - 2\,h = 0$
2.4. a)	$c\mu^3 h''' + b\mu h^2 h' + a\mu hh' + (d\mu - \lambda)h' = 0$
2.4. b)	$3\,c\,h''' + 3\,b\,h^2\,h' - h'\,z - h = 0$

where $\mathbf{u}^{(l)}(x)$ denotes the set of l-th partial derivatives of u.

The basic idea of the method is as follows. Equation (12) is augmented with the invariance surface condition

$$\Psi \equiv \sum_{i=1}^{p} \xi_i(x,u)\frac{\partial u}{\partial x_i} - \eta(x,u) = 0, \tag{13}$$

which is associated with the vector field

$$\mathbf{v} = \sum_{i=1}^{p} \xi_i(x,u)\partial_{x_i} + \eta(x,u)\partial_u. \tag{14}$$

The infinitesimal invariance criterion for equation (12) along with the invariant surface condition (13) imply that

$$\mathrm{pr}^{(s)}\mathbf{v}(\Delta)\big|_{\Delta=0,\Psi=0} = 0, \qquad \mathrm{pr}^{(s)}\mathbf{v}(\Psi)\big|_{\Delta=0,\Psi=0} = 0, \tag{15}$$

where $pr^{(s)}\mathbf{v}$ is the s-th prolongation of the vector field (14). This yields an overdetermined nonlinear system of equations for the infinitesimals.

To calculate the determining equations we implement the algorithm described in [9]. The application of this algorithm involves tedious, mechanical computations. Therefore, we make use of the Macsyma program *symmgrp.max* [8]. We can distinguish two different cases: $\tau \neq 0$ and $\tau = 0$.

3.1 Case 1: $\tau \neq 0$

In the case that $\tau \neq 0$, without loss of generality, we may set $\tau(x,t,u) = 1$, and we obtain a set of nine determining equations for the infinitesimals $\xi(x,t,u)$ and $\eta(x,t,u)$

$$\begin{aligned}
\xi_u &= 0,\\
\xi_{uu} &= 0,\\
\xi_{uuu} &= 0,\\
\eta_{uu} - 3\xi_{ux} &= 0,\\
\eta_{ux} - \xi_{xx} &= 0,\\
\eta_{uuu} - 3\xi_{uux} &= 0,\\
b\xi_u u^{2n} + a\xi_u u^n + c\eta_{uux} - c\xi_{uxx} - \xi\xi_u + d\xi_u &= 0,\\
b\eta_x u^{2n} + a\eta_x u^n - e\eta_u u + 3e\xi_x u + c\eta_{xxx} + d\eta_x - f\eta_u + \eta_t &\\
+3\xi_x\eta + e\eta + 3\xi_x f &= 0,\\
2b\xi_x u^{2n+1} + 2a\xi_x u^{n+1} + 2bn\eta u^{2n} + an\eta u^n + 4e\xi_u u^2 + 3c\eta_{uxx}u &\\
+3\xi_u\eta u + 4\xi_u fu - c\xi_{xxx}u - 3\xi\xi_x u + 2d\xi_x u - \xi_t u &= 0.
\end{aligned} \tag{16}$$

Solving this system for $e = f = 0$, we obtain:

1. If $n \neq \frac{1}{2}, 1$, $a = 0$
$$\xi = \frac{cx + 2cdt + 3k_1 + c}{3c(t+k_1)}, \qquad \eta = -\frac{u}{3n(t+k_1)}.$$
2. If $n = \frac{1}{2}$, $a = 0$,
$$\xi = \frac{cx + 2cdt + 3k_1 + c}{3c(t+k_1)}, \qquad \eta = -\frac{2u}{3(t+k_1)}.$$
3. If $n = 1$, $b \neq 0$,
$$\xi = \frac{2bcx + 4bcdt - a^2ct + 2bc + 6k_1 b}{6bc(t+k_1)}, \qquad \eta = \frac{-2bu - a}{6b(t+k_1)}.$$

In the above cases, k_1 represents an arbitrary constant.

3.2 Case 2: $\tau = 0$

In the case $\tau = 0$, without loss of generality, we may set $\xi = 1$ and the determining equation for the infinitesimal η are:

$$\begin{aligned}
&b\eta_x u^{2n+1} + a\eta_x u^{n+1} + 2bn\eta^2 u^{2n} + an\eta^2 u^n - e\eta_u u^2 + c\eta_{xxx}u + 3c\eta\eta_{uu}\eta_x u\\
&+3c\eta_{ux}\eta_x u + d\eta_x u + c\eta^3\eta_{uuu}u + 3c\eta^2\eta_{uux}u + 3c\eta^2\eta_u\eta_{uu}u + 3c\eta\eta_{uxx}u\\
&+3c\eta\eta_u\eta_{ux}u - f\eta_u u + \eta_t u + e\eta u = 0.
\end{aligned} \tag{17}$$

The complexity of this equation is the reason why we cannot solve (17) in general. Thus, we proceed by making ansatz on the form of $\eta(x,t,u)$.

For $n = \frac{1}{2}$, $a = 0$, $b = k$ or $n = 1$, $a = k$, $b = 0$, $k \neq 0$ an arbitrary constant, choosing $\eta = \eta(t)$, we find that the infinitesimal generator takes the form:

$$\xi = 1, \ \ \tau = 0, \ \ \eta = \frac{e}{k_1 \exp(et) - k}, \tag{18}$$

where k_1 is an arbitrary constant.

It is easy to check that generator (18) does not satisfy the Lie classical determining equations. From (18) the similarity variables for equation (1) have the form

$$z = t, \;\; u = \frac{e\,(x + h(z))}{k_1 \exp(et) - k}, \tag{19}$$

where $h(z)$ satisfies the following ODE

$$e\,h' + k_1\,f\,\exp(e\,z) - k\,f + d\,e = 0. \tag{20}$$

By solving this equation and substituting (19) we obtain the following exact solution of equation (1)

$$u(x,t) = \frac{e^2 x - k_1\,f\,\exp(e\,t) + \left(e\,f\,k - d\,e^2\right)\,t + k_2\,e^2}{e\,(k_1\,\exp(e\,t) - k)},$$

where k_2 represents an arbitrary constant.

4 Travelling wave solutions

Let us remember that, if we consider a, b, c, d, e, f and n arbitrary constants, we get the following generator

$$\lambda V_1 + \mu V_2 = \lambda \partial_x + \mu \partial_x. \tag{21}$$

We substitute (21) into the invariant surface condition

$$\eta(x,t,u) - \xi(x,t,u)\frac{\partial u}{\partial x} - \tau(x,t,u)\frac{\partial u}{\partial t} = 0, \tag{22}$$

and we obtain the similarity variable and the similarity solution

$$z = \mu x - \lambda t, \qquad u(x,t) = h(z). \tag{23}$$

Substituting (23) into (1) we obtain

$$c\mu^3 h''' + b\mu h^{2n} h' + a\mu h^n h' + d\mu h' - \lambda h' + eh + f = 0. \tag{24}$$

In order to obtain travelling wave solutions, we apply the simplest method to equation (24) with $n = 1$

$$c\mu^3 h''' + b\mu h^2 h' + a\mu h h' + d\mu h' - \lambda h' + eh + f = 0. \tag{25}$$

We assume that equation (25) has a solution in the following form

$$h(z) = a_0 + a_1 Y + \cdots + a_N Y^N, \tag{26}$$

where a_n $(n = 0,1,\ldots,N)$ are constant to be determined and $Y(z)$ is the general solution of the Riccati equation:

$$Y'(z) + Y^2(z) - \alpha Y(z) - \beta = 0, \tag{27}$$

with α and β unknown constants which must be determined. Taking the homogeneous balance between the highest order derivative h''' and the nonlinear term of highest order $h^2 h'$ we obtain $N = 2$. Therefore, the solution of (25) takes the following form

$$h(z) = a_0 + a_1 Y + a_2 Y^2. \tag{28}$$

Inserting (28) and the derivatives h', h'', ..., into (25) we get a polynomial in $Y(z)$ and its derivatives. Requiring the vanishing of the coefficients of the different powers of the function $Y(z)$, we obtain an overdetermined system of equations

$$\begin{aligned}
&aa_0a_1\beta\mu + aa_0a_2\beta\mu + a_0{}^2a_1b\beta\mu + a_0{}^2a_2b\beta\mu \\
&+a_0e + a_1c\alpha^2\beta\mu^3 - 2a_1c\beta^2\mu^3 + a_1d\beta\mu - a_1\beta\lambda + a_2c\alpha^2\beta\mu^3 \\
&-2a_2c\beta^2\mu^3 + a_2d\beta\mu - a_2\beta\lambda + f = 0,
\end{aligned}$$

$$\begin{aligned}
&aa_0a_1\alpha\mu + aa_0a_2\alpha\mu + aa_1{}^2\beta\mu + 2aa_1a_2\beta\mu \\
&+aa_2{}^2\beta\mu + a_0{}^2a_1b\alpha\mu + a_0{}^2a_2b\alpha\mu + 2a_0a_1{}^2b\beta\mu + 4a_0a_1a_2b\beta\mu \\
&+2a_0a_2{}^2b\beta\mu + a_1c\alpha^3\mu^3 - 8a_1c\alpha\beta\mu^3 + a_1d\alpha\mu + a_1e - a_1\alpha\lambda + a_2c\alpha^3\mu^3 \\
&-8a_2c\alpha\beta\mu^3 + a_2d\alpha\mu + a_2e - a_2\alpha\lambda = 0,
\end{aligned}$$

$$\begin{aligned}
&-aa_0a_1\mu - aa_0a_2\mu + aa_1{}^2\alpha\mu + 2aa_1a_2\alpha\mu \\
&+aa_2{}^2\alpha\mu - a_0{}^2a_1b\mu - a_0{}^2a_2b\mu + 2a_0a_1{}^2b\alpha\mu + 4a_0a_1a_2b\alpha\mu + 2a_0a_2{}^2b\alpha\mu \\
&+a_1{}^3b\beta\mu + 3a_1{}^2a_2b\beta\mu + 3a_1a_2{}^2b\beta\mu - 7a_1c\alpha^2\mu^3 + 8a_1c\beta\mu^3 - a_1d\mu + a_1\lambda \\
&+a_2{}^3b\beta\mu - 7a_2c\alpha^2\mu^3 + 8a_2c\beta\mu^3 - a_2d\mu + a_2\lambda = 0,
\end{aligned}$$

$$\begin{aligned}
&12a_2c\alpha\mu^3 - aa_1{}^2\mu - 2aa_1a_2\mu - aa_2{}^2\mu - 2a_0a_1{}^2b\mu - 4a_0a_1a_2b\mu \\
&-2a_0a_2{}^2b\mu + a_1{}^3b\alpha\mu + 3a_1{}^2a_2b\alpha\mu + 3a_1a_2{}^2b\alpha\mu + 12a_1c\alpha\mu^3 + a_2{}^3b\alpha\mu = 0.
\end{aligned}$$

$$\mu(-(a_1+a_2))\left(a_1{}^2b + 2a_1a_2b + a_2{}^2b + 6c\mu^2\right) = 0. \tag{29}$$

Solving (29) we obtain that this system is satisfied when

$$a_2 = -a_1 \qquad a_0 = -\frac{f}{e}. \tag{30}$$

Equation (27) with $\beta = 0$ is the Bernoulli equation, so we can obtain the corresponding solution h of the ODE (25) in terms of this equation. As a result, the solution of the Bernoulli equation is

$$Y(z) = \alpha\left(\frac{Y_1+Y_2}{1+Y_1+Y_2}\right), \tag{31}$$

where $Y_1(z) = \sinh(\alpha(z+\delta))$, $Y_2(z) = \cosh(\alpha(z+\delta))$ and δ is an arbitrary constant. Substituting (31) into (28) we obtain the following solution

$$h(z) = \frac{f}{e} + \frac{\alpha}{2}\left[a_1\left(1+\tanh\left(\frac{\kappa}{2}(z+\delta)\right)\right) - 2\,\alpha\,a_1\tanh\left(\frac{\alpha}{2}(z+\delta)\right)\right].$$

By using transformation (23) we can obtain a solution of equation (1) with $n = 1$.

5 Conclusions

In this paper, we have studied the classical and nonclassical symmetries admitted by the generalized Gardner equation (1). We have established a symmetry classification of equation (1) in terms of the arbitrary constants

a, b, c, d, e, f and n. We have proved that the nonclassical method applied to this equation leads to new symmetries. Furthermore, similarity variables and similarity solutions of equation (1) have been obtained from the optimal system of subalgebras. Taking into account the elements of the optimal system of subalgebras we reduce equation (1) to an ordinary differential equation. Finally, we have constructed some travelling wave solutions by using the modified simplest equation method.

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