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Multiple solutions of the Kirchhoff-type problem in R^N

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Abstract

In this paper, we concern with a class of quasilinear Kirchhoff-type problem. By using the Ekeland's Variational Principle and Mountain Pass Theorem, the existence of multiple solutions is obtained. Besides, we also take this problem as an example to give the main frame of using critical point theory to find the weak solutions of nonlinear partial differential equation.

Keywords: Mountain Pass Theorem; Ekeland's Variational Principle; Kirchhoff-type problem; (PS) sequence
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1 Introduction

This paper is to investigate the existence of multiple solutions for the following Kirchhoff-type problem

$$\left(1+b\int_{R^N}(|\nabla u|^2+V(x)u^2)dx\right)[-\triangle u+V(x)u]=f(x,u)\ , \quad \text{in } R^N \tag{1}$$

where $N=1,2,3$, constant $b>0$. Set $H:=H^1(\mathbb{R}^N)=\{u\in L^2(\mathbb{R}^N)\ \big|\ \nabla u\in L^2(\mathbb{R}^N)\}$ with the inner product $(u,v)_H=\int_{\mathbb{R}^N}(\nabla u\nabla v+uv)dx$ and the norm $\|u\|_H=(u,u)_H^{\frac{1}{2}}$.

In order to obtain that the variational structure of (1) and the corresponding variational functional I belongs to $C^1(H,R)$, we need the some basic hypotheses such that $f(x,u)\in C(R^N\times R,R)$ and $V\in C(R^N,R)$, f is superlinear

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near origin, subcritical growth; V has a positive infimum. Then, we can replace seeking the weak solutions of (1) with finding the critical points of the variational functional I (i.e. solving the Euler-Lagrange equations (1) of I).

Under these basic hypotheses, if set $F(x,u) := \int_0^u f(x,s)ds$, the variational functional of (1) is

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_{R^N} F(x,u)dx \tag{2}$$

and it is of class C^1(see Lemma 4) on the real Sobolev space $E := \{u \in H \mid \int_{R^N} V(x)u^2dx < +\infty\}$, which is equipped with the inner product and corresponding norm

$$(u,v) = \int_{R^N} [\,\nabla u \nabla v + V(x)uv\,]\,dx, \quad \|u\| = (u,u)^{\frac{1}{2}}.$$

Concretely speaking, we suppose that f and V satisfy the following assumptions:

(V_1) $\inf_{x\in R^N} V(x) \geq V_0 > 0$, and for any $M > 0, meas\{x \in R^N : V(x) \leq M\} < +\infty$ where V_0 is a positive constant and *meas* denote the Lebesgue measure in R^N.

(f_1) $f(x,u) = o(u)$ uniformly in x as $|u| \to 0$.

(f_2) $|f(x,u)| \leq c(1+|u|^{p-1})$ for some $c > 0$ and $p \in [2,2^*)$, where $2^* = \begin{cases} \frac{2N}{N-2}, \ N \geq 3; \\ +\infty, \ N = 1, \ 2. \end{cases}$

(f_3) There exist $\mu > 4$ and $r > 0$ such that

$$0 < \mu F(x,u) \leq uf(x,u) \tag{3}$$

for all $u \in R \setminus \{0\}$ and $x \in R^N$ and

$$F_r := |r|^{-\mu} \inf_{x\in R^N,\ |u|=r} F(x,u) > 0.$$

(f_3') $F(x,u) \leq \frac{1}{4}uf(x,u) \ \ \forall x \in R^N, \forall u \in R.$

(f_4) $\frac{F(x,u)}{u^4} \to +\infty$ uniformly in x as $|u| \to +\infty$.

(f_5) $u \mapsto \frac{f(x,u)}{|u|^3}$ is strictly increasing on $(-\infty,0) \cup (0,+\infty)$.

Set $V(x) \equiv 0$ and replace R^N by a bounded smooth domain $\Omega \subseteq R^N$ respectively, (1) reduces to the following Dirichlet problem of Kirchhoff type

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2dx)\triangle u = f(x,u), & in \ \ \Omega; \\ u = 0, & on \ \partial\Omega. \end{cases} \tag{4}$$

(4) is related to the stationary analogue of the equation

$$u_{tt} - (a+b\int_\Omega |\nabla u|^2dx)\Delta u = g(x,u) \tag{5}$$

which was proposed by Kirchhoff [2] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. The Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. It is pointed in [3] that (4) model several physical and biological systems, where u describes a process which depends on the average of itself (for example, population density). (5) received much attention only after Lions [4] introduced an abstract framework to it. Some interesting studies of (4) by variational methods can be found in [2, 3, 5–10].

Besides, (1) is variant of the following problem

$$-(a+b\int_{R^N}|\nabla u|^2dx)\triangle u+V(x)u=f(x,u),\quad x\in R^N. \tag{6}$$

Recently, problems like (1)(6) have been extensively studied by minimax theory in critical point theory, e.g. Mountain Pass Theorem [11] and Symmetric Mountain Pass Theorem [12], Linking Theorem and Fountain Theorem [13] and its variants [14] (see for example [15–18] for (6)).

Recently, some authors have studied (1) with perturbation methods(refer to [19, 20, 22, 23]). Particularly, Azzollini, d′Avenia and Pomponio [19] given a perturbation method and used it to obtain the multiple radial symmetric solutions of (1) with $V(x)\equiv 0$. Alves and Figueiredo [20] obtained two positive solutions of (1) by Nehari method, where $V(x)$ is periodic, nonlocal term $M(\int_{R^N}(|\nabla u|^2+V(x)u^2)$ is general and $f(x,u)$ is subcritical or critical growth. By a general perturbed theorem in [21], Li, Li and Shi [22] got a positive solution of (1) provided that $V(x)\equiv\lambda\geq 0$ and $f(x,u)$ is independent of x combining cut-off functional with monotonicity trick. Ji [23] established the existence of infinitely many radially symmetric solutions of problem $p(x)$-(1) with radial potential via a direct variational method and the principle of symmetric criticality.

However, as far as we known, little has been done for the existence and multiplicity of nontrivial solutions of (1). Motivated by the above facts, this paper is to study the existence and multiplicity of nontrivial solutions of (1) by combining the direct method of the calculus of variation with minimax theory in critical point theory. Concretely, we shall find two distinct critical values of I: (1) a negative critical value of I natural constrained in a neighborhood of zero via Ekeland's Variational Principle is obtained and (2) a positive critical value of I is obtained via the Mountain Pass Theorem.

Our results is as follows:

Theorem 1. *If conditions $(V_1)(f_1)(f_2)(f_3)$ holds, then the problem (1) has a positive and a negative solution.*

Theorem 2. *If condition $(f_3')(f_4)$ is used in place of (f_3), then the conclusions of Theorem 1 holds still.*

Theorem 3. *If condition (f_5) is used in place of (f_3'), then the conclusions of Theorem 2 holds still.*

Remark 1. (V_1) is weaker than the coercivity of V, namely $V(x)\to+\infty, as|x|\to+\infty$.

Remark 2. $(f_1)(f_5)$ not only implies (f_3'), but also that $F(x,u)>0$ holds for $u\neq 0$.

The remainder of this paper is organized as follows. Section 2 presents some preliminary results. Section 3 is devoted to the proof of results. Through out the paper, c_i and c are used in various places to denote positive constants.

2 Main frame of the corresponding method

Besides the multiple results of (1), we also take (1) as an example to give the main frame of finding the critical points of the variational functional I. The method of sequence convergence is a powerful tool to find the critical points of I by variational methods, the core idea (see [11, 13]) of which is essential as following:

Step 1. **To obtain the variational functional I of (1) and prove that $I\in C^1(H,R)$.**
By Theorem 7.7 in [24], (1) has a variational functional I such as (2). In order to prove that $I\in C^1(H,R)$, it is necessary to assume that $f(x,u)\in C(R^N\times R,R)$ and there is a positive constant c such that

$$|f(x,u)|\leq c(|u|+|u|^{p-1}), p\in[2,2^*] \tag{7}$$

and $V\in C(R^N,R)$ has a positive infimum V_0. (f_1) is often called that $f(x,u)$ is superlinear near origin, (f_2) is often referred as that $f(x,u)$ is subcritical growth, (f_4) is also known as $f(x,u)$ is 3-superlinear at ∞ and (3) is the famous (AR) condition. In fact, it is easy to see that $(f_1)(f_2)$ implies (7).

Step 2. **To structure a good candidate for being a critical value of I.**
$(f_1)(f_2)$(3) has been often used to show that I has a mountain pass geometry, namely that there are different points u_1, u_2 in $H^1(\mathbb{R}^N)$ such that

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)) > \max\{I(u_1), I(u_2)\},$$

where $\Gamma := \{\gamma \in C([0,1], H) | \gamma(0) = u_1, \gamma(1) = u_2\}$, then the mountain pass level c is a good candidate for being a critical value of I. Furthermore, by the Deformation Lemma or the Ekeland's Variational Principle, one sees that the mountain pass geometry directly implies the existence of a sequence $\{u_n\}$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$. Such a sequence is called a Palais-Smale sequence at c ($(PS)_c$ sequence for short) .

In order to complete the first two steps, assumptions on $f(x,u)$ should be $(f_1)(f_2)$ and (f_4) rather than (7) and (3). In fact, $(f_1)(f_2)(f_4)$ are most essential because (3) implies (f_4).

Step 3. **To prove that $(PS)_c$ condition holds, namely any $(PS)_c$ sequence $\{u_n\}$ possess a convergent subsequence.**
In general, to establish $(PS)_c$ condition, it suffices to insure the boundedness of $(PS)_c$ sequence if I is weakly lower semicontinuous on H, $I'(\cdot) : H \to H^*$ is compact and H is reflexive. In this process, (3) is also often used to obtain the boundedness of the $(PS)_c$ sequence, and (f_2) is also often combined with the following compactness of the embedding

$$H \hookrightarrow L^p(R^N), p \in [2, 2^*) \tag{8}$$

to prove that $I'(\cdot) : H \to H^*$ is compact and I is weakly lower semicontinuous.

Step 4. **By the continuity of I and I', I has a critical point u_0 at c.**

To study (1) by using this method, there is general two difficulties.

One is that the dimension N must be restricted less than or equal to 3 if f satisfies $(f_2)(f_4)$. Under (3), several researchers studied (6). He and Zou [15] studied the existence, multiplicity and concentration behavior of positive solutions of (6) with $N = 3$ by using the variational methods. Wu [16] obtained the nontrivial solutions and a sequence of high energy solutions of (6) with $N = 1,2,3$ by using symmetric mountain pass theorem. Without (3), Liu and He [17] obtained infinitely many high energy solutions of (6) with $N = 3$ and sublinear $f(x,u) = (p+1)b(x)|u|^{p-1}u (0 < p < 1)$ combined a variant version of Fountain Theorem [14] and symmetric mountain pass Theorem to the Schrödinger operator eigenvalue theory.

Another is the lack of compactness of the embedding (8). Traditionally this difficult can be avoided by two class techniques. One is that, if $V(x)$ satisfies (V_1), the compactness of embedding is regained by restricting the working space H to the subspace E. Another is that, if $V(x) \equiv V_0 > 0$ and $f(x,u)$ depend only on $|x|$ (namely which are radially symmetric), the compactness of embedding can be regained by restricting to the subspace H_r also. Jin and Wu [18] dealt with (6) with $V(x) \equiv 1$ in H_r, obtained infinitely many radial solutions. Moreover, if $V(x)$ and $f(x,u) = Q(x)|u|^{p-1}$ are radially symmetric and satisfies some conditions in [25], the compactness of embedding of some weighted Sobolev spaces

$$H_r^1(R^N; V) \hookrightarrow L^p(R^N; Q)$$

is ensured also.

3 Preliminary lemmas

Let $\|u\|_p$ be the usual norm of the Lebesgue space $L^p(R^N)$. By Sobolev imbedding theorem (see [13, Theorem 1.8], the following embedding

$$H \hookrightarrow L^p(R^N), p \in [2, 2^*) \tag{9}$$

is continuous, that is, there are positive constants ν_p such that

$$\|u\|_p \le \nu_p \|u\|_H, \forall u \in H.$$

By (V_1), the embedding

$$E \hookrightarrow H \tag{10}$$

is continuous, that is, there is positive constant $c_{V_0} = \frac{1}{\sqrt{\min\{1,V_0\}}}$ such that

$$\|u\|_H \le c_{V_0}\|u\|, \forall u \in E.$$

By Lemma 3.4 in [27], (V_1) implies that the embedding

$$E \hookrightarrow L^p(R^N), p \in [2,2^*) \tag{11}$$

is compact.

Lemma 4. *If $(V_1)(f_1)(f_2)$ hold, then $I \in C^1(E,R)$, I is weakly lower semicontinuous on E and*

$$\langle I'(u), v\rangle = (u,v) + b(u,u)(u,v) - \langle \Psi'(u), v\rangle. \tag{12}$$

Moreover, $I'(\cdot): E \to E^*$ is compact, where $\Psi(u) = \int_{R^N} F(x,u)dx$.

Proof: To begin with, according to the fact that $\|\cdot\|: E \to [0,+\infty)$ is C^1, it suffices to show that $\Psi(u)$ is C^1 and I is weakly lower semicontinuous.
(i) To verify $\Psi \in C^1(E,R)$.
By (f_1), for any $\varepsilon > 0$ given, there is a constant $\delta_\varepsilon > 0$ such that

$$|f(x,u)| \le \varepsilon|u|, \text{ for all } x \in R^N \text{ and } |u| \le \delta_\varepsilon.$$

By (f_2), there is constant $c_\varepsilon := (1 + \frac{1}{\delta_\varepsilon^{p-1}})$ such that

$$|f(x,u)| \le c_\varepsilon |u|^{p-1}, \text{ for all } x \in R^N \text{ and } |u| \ge \delta_\varepsilon.$$

Combining two estimates above, there is that

$$|f(x,u)| \le \varepsilon|u| + c_\varepsilon|u|^{p-1} \tag{13}$$

and

$$|F(x,u)| \le \frac{\varepsilon}{2}|u|^2 + \frac{c_\varepsilon}{p}|u|^p, \tag{14}$$

for all $x \in R^N$ and for all $u \in R$.
By (9) and (10), there are constants $\eta_p := c_{V_0}\nu_p > 0$ such that

$$\|u\|_p \le \eta_p\|u\|, \quad \forall u \in E. \tag{15}$$

In view of (13)-(15) and Hölder inequality, it is well known (see [13, Lemma 3.10]) that $\Psi \in C^1(E,R)$ and

$$\langle \Psi'(u), v\rangle = \int_{R^N} f(x,u)v dx, \forall u, v \in E.$$

(ii) To verify I is weakly lower semicontinuous on E.
Let $u_n \rightharpoonup u$ in E, then $\{u_n\}$ is bounded in E . Along a subsequence, (11) yields

$$u_n \to u \text{ in } L^p(R^N), p \in [2,2^*) \tag{16}$$

which implies that functional $\Psi(u)$ is weakly continuous on E(see [16, Lemma 3]).

Fixed $u \in E$, let $\{u_n\} \subseteq E$ and $u_n \rightharpoonup u$ in E. Note that the following functional defined by inner product

$$(u, \cdot) : E \to R$$

is bounded linear functional, that is, $(u, \cdot) \in E^*$. Hence, $(u, u_n) \to (u, u)$ as $n \to +\infty$. It follows from the Cauchy-Schwarz inequality that

$$\|u\| \leq \liminf_{n \to +\infty} \|u_n\|.$$

Hence,

$$\begin{aligned}
I(u) &= \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \Psi(u) \\
&\leq \frac{1}{2}\left(\liminf_{n \to +\infty} \|u_n\|\right)^2 + \frac{b}{4}\left(\liminf_{n \to +\infty} \|u_n\|\right)^4 - \liminf_{n \to +\infty} \Psi(u_n) \\
&\leq \frac{1}{2}\liminf_{n \to +\infty} \|u_n\|^2 + \frac{b}{4}\liminf_{n \to +\infty} \|u_n\|^4 - \liminf_{n \to +\infty} \Psi(u_n) \\
&= \liminf_{n \to +\infty} I(u_n).
\end{aligned}$$

So, I is weakly lower semicontinuous on E.

Next, to prove that $I'(\cdot) : E \to E^*$ is compact. Indeed, if $u_n \subset E$ is bounded, then passing a subsequence, one has that $u_n \rightharpoonup u$ in E and $u_n \to u$ in $L^p(R^N), p \in [2, 2^*)$. By the Hölder inequality and (12), Theorem A.4 in [13] implies

$$\begin{aligned}
\|\Psi'(u_n) - \Psi'(u)\| &= \sup_{\|v\| \leq 1} \left| \int_{R^N} [f(x, u_n) - f(x, u)] v dx \right| \\
&\leq \sup_{\|v\| \leq 1} \int_{R^N} |f(x, u_n) - f(x, u)||v| dx \to 0
\end{aligned}$$

as $n \to \infty$. Note that

$$\langle I'(u_n) - I'(u), v\rangle = (u_n - u, v) + b[(u_n, u_n)(u_n, v) - (u, u)(u, v)] - [\langle \Psi'(u_n) - \langle \Psi'(u), v\rangle].$$

Hence, $I'(\cdot) : E \to E^*$ is compact. □

The following lemma shows that I has a mountain pass geometry on E:

Lemma 5. *Assume that $(V_1)(f_1)(f_2)(f_3)$ hold, there exist constants $\rho, \alpha > 0$ and $v \in E$ with $\|v\| < \rho$ such that*

$$I(0) = 0, \quad I|_{\partial B_\rho(0)} \geq \alpha, \quad \text{and} \quad I(v) < 0, \; \text{where} \; \partial B_\rho(0) := \{u \in E \mid \|u\| = \rho\}.$$

Proof. *(1)* To verity that 0 is a local minimum of I.

Obviously, $I(0) = 0$. Moreover, for any $\varepsilon \in (0, \frac{1}{2\eta_2^2})$($\eta_2$ appear in (15)), in view of (2) and (14)(15), there holds

$$I(u) \geq \frac{1 - \varepsilon\eta_2^2}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{c_\varepsilon \eta_p^p}{p}\|u\|^p \geq \|u\|^2\left(\frac{1}{4} - \frac{c_\varepsilon \eta_p^p}{p}\|u\|^{p-2}\right).$$

Hence, by fixing $\rho \in (0, (\frac{p}{4c_\varepsilon \eta_p^p})^{\frac{1}{p-2}})$, it is easy to see such that

$$I|_{\partial B_\rho(0)} \geq \alpha := \rho^2\left(\frac{1}{4} - \frac{c_\varepsilon \eta_p^p}{p}\rho^{p-2}\right) > 0.$$

(2) (f_3) implies $f(x, u)$ is 3-superlinear at ∞.

Indeed, let $r > \rho$, for any $(x,u) \in R^N \times R$, set

$$g(t) := t^{-\mu} F(x,tu), \quad t \in [r|u|^{-1}, 1].$$

By (f_3), for any $|u| \geq r$ and any $t \in [r|u|^{-1}, 1]$, we have that

$$g'(t) = t^{-\mu-1}[f(x,tu)tu - \mu F(x,tu)] \geq 0.$$

So, $g(1) \geq g(r|u|^{-1})$, that is,

$$F(x,u) \geq r^{-\mu} F(x, ru|u|^{-1})|u|^{\mu} \geq F_r |u|^{\mu}, \quad \forall |u| \geq r, \forall x \in R^N, \tag{17}$$

where constant $F_r > 0$ is given in (f_3).

Choosing any $u_0 \in L^{\mu}(R^N)$ with $\|u_0\|_{\mu} = 1$. Then,

$$I(tu_0) = \frac{t^2}{2}\|u_0\|^2 + \frac{bt^4}{4}\|u_0\|^4 - \int_{R^N} F(x,tu_0)dx \leq t^2\Big(\frac{\|u_0\|^2}{2} + \frac{bt^2\|u_0\|^4}{4} - F_0 t^{\mu-2}\Big) \to -\infty$$

as $t \to +\infty$. Then, we can choose $v := t_1 \eta_{\mu} u_0 \in E$ with $t_1 > \rho$ such that

$$I(v) < 0. \tag{18}$$

(3) Let c be the mountain pass level of I such that

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

where $\Gamma := \{\gamma \in C([0,1], E) \mid \gamma(0) = \theta, \gamma(1) = v\}$. For any $\gamma \in \Gamma$, it is clear that functional

$$\|\gamma(\cdot)\| : [0,1] \to [0, \|v\|]$$

is continuous. By intermediate value theorem, there exists a point $t \in [0,1]$ at least such that $\|\gamma(t)\| = \rho$, that is,

$$\gamma([0,1]) \cap \partial B_{\rho}(0) \neq \emptyset.$$

So, we can choose $u_1 \in \gamma([0,1]) \cap \partial B_{\rho}(0)$ such that

$$I(u_1) \geq \inf_{u \in \partial B_{\rho}(0)} I(u) \geq \alpha > 0 = \max\{0, I(v)\}.$$

Hence, I has a mountain pass structure at c on E. $\square$

Lemma 6. *If $\{u_n\}$ is a bounded $(PS)_c$ sequence of I, then it has a convergent subsequence.*

Proof. Let $\{u_n\} \subseteq E$ be a bounded $(PS)_c$ sequence. Going if necessary to a subsequence, by the reflexivity of E and (11), we have that $u_n \rightharpoonup u, in\ E$ and (16). By (12), there is that

$$\begin{aligned}(1 + b\|u_n\|^2)\|u_n - u\|^2 &= \langle I'(u_n) - I'(u), u_n - u\rangle + b(\|u\|^2 - \|u_n\|^2)(u_n - u, u) \\ &+ \int_{R^N} [f(x,u_n) - f(x,u)](u_n - u)dx\end{aligned} \tag{19}$$

Note that $I'(u) \in E^*$ and $I'(u_n) \to 0$, it is easy to get that

$$|\langle I'(u_n) - I'(u), u_n - u\rangle| \leq |\langle I'(u_n), u_n - u\rangle| + |\langle I'(u), u_n - u\rangle| \to 0 \tag{20}$$

as $n \to +\infty$.

By the fact that $(u,\cdot) \in E^*$ and boundedness of sequence $\{u_n\}$, there is that

$$b|(\|u\|^2 - \|u_n\|^2)(u_n - u, u)| \to 0 \tag{21}$$

as $n \to +\infty$. By (13) and the Hölder inequality, we have that

$$\begin{aligned} &|\int_{R^N} [f(x,u_n) - f(x,u)](u_n - u)dx| \\ \leq &\int_{R^N} [\varepsilon(|u| + |u_n|) + C_\varepsilon(|u|^{p-1} + |u_n|^{p-1})] \cdot |u_n - u| dx \\ \leq &\ \varepsilon(\|u\|_2 + \|u_n\|_2)\|u_n - u\|_2 + C_\varepsilon(\|u\|_p^{p-1} + \|u_n\|_p^{p-1})\|u_n - u\|_p. \end{aligned} \tag{22}$$

By (16), there is that

$$\int_{R^N} [f(x,u_n) - f(x,u)](u_n - u)dx \to 0 \tag{23}$$

as $n \to +\infty$.

Hence, by (19)-(23) and the fact that $I'(\cdot) : E \to E^*$ is compact (see Lemma 4), one has that $\|u_n - u\| \to 0$ as $n \to +\infty$. □

4 Proof of results

4.1 Proof of Theorem 1

First of all, by Lemma 4, we have that $I \in C^1(E,R)$.

Then, we can find a minimization of I constrained in the bounded closed subset $(\overline{B}_{t_1}, \|\cdot\|)$ of E, where t_1 is given in (18) and

$$\overline{B}_{t_1} := \{u \in E \mid \|u\| \leq t_1\}.$$

Indeed, by (11)(14), I is bounded on $\overline{B}_{t_1}$. Hence,

$$-\infty < c_0 = \inf_{u \in \overline{B}_{t_1}} I(u) < 0.$$

It is easy to see that $(\overline{B}_{t_1}, d)$ is complete metric space with the metric d defined by

$$d(x,y) := \|x - y\|, \forall x, y \in \overline{B}_{t_1}.$$

By Ekeland's Variational Principle, functional I has a minimizing sequence $\{u_n\} \subseteq \overline{B}_{t_1}$ such that $I(u_n) \to c_0, I'(u_n) \to 0 \ as \ n \to +\infty$. □

Note the fact that $\overline{B}_{t_1}$ is closed convex set in E, the reflexivity of space E implies that $\overline{B}_{t_1}$ is weakly compact in E. So, going if necessary to a subsequence, $u_n \rightharpoonup w_1 \in \overline{B}_{t_1}$. By Lemma 4, the weak lower semicontinuity of I implies that

$$c_0 = \lim_{n \to +\infty} I(u_n) = \liminf_{n \to +\infty} I(u_n) \geq I(w_1) \geq c_0.$$

Therefore, $I(w_1) = c_0 < 0, I'(w_1) = 0$.

Next, by Lemma 5, I has a mountain pass structure at

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

which implies that I has a $(PS)_c$ sequence via Ekeland's Principle or Deformation Theorem(see [28]), namely

$$I(u_n) \to c > 0, I'(u_n) \to 0.$$

Moreover, I satisfies $(PS)_c$ condition. In fact, in view of Lemma 6, it suffices to check that $\{u_n\}$ is bounded. If not, by (f_3), for n large enough, there is that

$$\begin{aligned}
c+1+\|u_n\| &\geq I(u_n)-\frac{1}{\mu}\langle I'(u_n),u_n\rangle \\
&= \frac{\mu-2}{2\mu}\|u_n\|^2+\frac{b(\mu-4)}{4\mu}\|u_n\|^4+\int_{R^N}[\frac{1}{\mu}f(x,u_n)u_n-F(x,u_n)]dx \\
&\geq \frac{\mu-2}{2\mu}\|u_n\|^2+\frac{b(\mu-4)}{4\mu}\|u_n\|^4. \qquad (24)
\end{aligned}$$

It is a contradiction.

Hence, by the weakly lower semicontinuity of I and the unique of the limits of sequences $\{I(u_n)\}\subseteq R$ and $\{I'(u_n)\}\subseteq E^*$, it is easy to obtain that $I'(u_n)\to I(u)=c$ and $I'(u_n)\to I(u)=0$, I has the second critical point $w_2\in E$ at mountain pass level c such that $I(w_2)=c>0, I'(w_2)=0$. □

4.2 Proof of Theorem 2

From the proof of Theorem 1, as it is obvious that (f_3') implies (24), it is suffices to prove that (f_4) implies (18). In fact, like (2.6) in [16], (f_4) implies that there is a point $e\in\widetilde{E}\setminus B_\rho$ on any finite dimensional subspace $\widetilde{E}\subset E$ such that $I(e)<0$. Therefore, (f_4) implies (18) similar to (3). □

4.3 Proof of Theorem 3

From the proof of Theorem 2, it is suffices to prove that (f_5) implies (f_3'). In fact, (f_5) implies that for any $u>0$,

$$F(x,u)=\int_0^1 f(x,ut)udt=\int_0^1\frac{f(x,ut)u^4t^3}{(ut)^3}dt\leq\int_0^1\frac{f(x,u)u^4t^3}{u^3}dt=\frac{1}{4}uf(x,u)$$

and for any $u<0$,

$$F(x,u)=\int_0^1 f(x,ut)udt=-\int_0^1\frac{f(x,ut)u^4t^3}{|ut|^3}dt\leq-\int_0^1\frac{f(x,u)u^4t}{|u|^3}dt=\frac{1}{4}uf(x,u).$$

Hence, (f_3') holds. □

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