On the perturbed restricted three-body problem

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Abstract

In this survey paper we offer an analytical study regarding the perturbed planar restricted three-body problem in the case that the three involved bodies are oblate. The existence of libration points and their linear stability are explored under the effects of the perturbations in Coriolis and centrifugal forces. The periodic orbits around these points are also studied under these effects. Moreover, the elements of periodic orbits around these points are determined.

1 Introduction

The three-body problem has a special relevance, particularly in astrophysics and astrodynamics. In general, the three-body problem is classified into two types:

(i) The general three-body problem, which describes the motion of three bodies of arbitrary masses under their mutual attraction due to the gravitational field. The motion of the bodies take place in three dimensions and there are no restrictions on their masses nor on the initial conditions.

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(ii) The restricted three-body problem in which the mass of the third body is very small in comparison with the masses of the primaries, and it does not affect their motion. In this case, the primaries move around their center of mass either along circular or elliptical orbits.

Thus, the general problem has some applications in celestial mechanics such as the dynamics of triple star systems (and only a very few in space dynamics and solar system dynamics), whereas the restricted problem plays an important role in studying the motion of artificial satellites. It can be used also to evaluate the motion of the planets, minor planets and comets. The restricted problem gives an accurate description not only regarding the motion of the Moon but also with respect to the motion of other natural satellites. Furthermore, the restricted problem has many applications not only in celestial mechanics but also in physics, mathematics and quantum mechanics, to name a few. In quantum mechanics, a general form of the restricted problem is formed to solve the Schrödinger equation of helium-like ions. Furthermore, in modern solid state physics, the restricted problem can be used to discuss the motion of an infinitesimal mass affected not only by the gravitational field but also by light pressure from one (or both) of the primaries, which is called the photogravitational problem.

In the last decades, the restricted three-body problem has been enhanced by a great number of researches. Many of them deal with the effects of the perturbed forces such as the lack of sphericity, photogravitational force, Coriolis and centrifugal forces, variation of masses, the Pointing-Robertson and Yarkovsky effects and the atmospheric drag and solar wind. The Kirkwood gaps in the ring of the asteroid's orbits lying between the orbits of Mars and Jupiter are examples of the perturbation produced by Jupiter on an asteroid. This motivates many researchers to study the restricted three-body problem under the effects of small perturbations in Coriolis and centrifugal forces when the three involved objects are oblate, ..., etc.

The significance of the restricted problem when describing actual physical situations can be judged by the results obtained when these are compared with observations. It is worth mentioning that the utility might be prejudged by order of magnitude evaluations regarding the masses and the distances of the participating bodies. In this way, a classic example in space dynamics is the Sun-Earth-Moon system.

Some important contributions related to the libration points in the restricted three-body problem with one or both primaries are oblate spheroids when the equatorial plane is coincident with the plane of motion were studied by Subbarao and Sharma [28], Sharma and Subbarao [23], and Markellos et al. [20]. Abouelmagd [9] also studied the effects of oblateness J_2 and J_4 for the more massive primary in the planar restricted three-body problem on the locations of the triangular points and their linear stability. He found that these locations are affected by the coefficients of oblateness. Furthermore, he investigated that the triangular points are stable for $0 < \mu < \mu_c$ and unstable when $\mu_c \leq \mu \leq 1/2$, where μ_c is the critical mass parameter which depends on the coefficients of oblateness.

The existence of libration points and their linear stability as well as periodic orbits around these points when the more massive primary is radiating and the smaller is an oblate spheroid were studied by Abouelmagd and Sharaf [3]. Their study also includes the effects of oblateness of $\bar{J}_{2i} : i = 1, 2$, with respect to the smaller primary in the restricted three-body problem.

The restricted problem when the three participating bodies are oblate spheroids was also studied by Elipe and Ferrer [15], and El-Shaboury and El-Tantawy [18]. When one or two of the primaries are triaxial bodies this problem was introduced by El-Shaboury et al. [17], Khanna and Bhatnagar [19], and Sharma et al. [24].

Various researchers made studies in the restricted three-body problem under the effects of small perturbations in centrifugal and Coriolis forces such as in Szebehely [29], Bhatnagar and Hallan [12], Devi and Singh [14], and Shu and Lu [25], to quote some of them.

The effect of small perturbations $\varepsilon, \varepsilon'$ in Coriolis and centrifugal forces with variable mass in the restricted three-body problem has been studied by Singh [26]. He found that in the nonlinear sense the triangular points are stable for all mass ratios in the range of linear stability except for three mass ratios which depend on $\varepsilon, \varepsilon'$ and the constant β due to the variation of mass governed by Jeans' law.

Mittal et al. [21] have studied periodic orbits generated by Lagrangian solutions of the restricted three-body problem when the bigger body is a source of radiation and the smaller is an oblate spheroid. They used

the definition of Karimov and Sokolsky for mobile coordinates to determine these orbits, they also used the predictor method to draw them.

Singh and Begha [27] have studied the existence of periodic orbits around the triangular points in the restricted three-body problem when the bigger primary is a triaxial and the smaller primary is considered as an oblate spheroid in the range of linear stability with the perturbed Coriolis and centrifugal forces. They deduced that long and short periodic orbits exist around these points and their periods, orientation and eccentricities are affected by the non sphericity and the perturbation in Coriolis and centrifugal forces.

The existence of libration points and their linear stability when the three participating bodies are oblate spheroids and the primaries are radiation source as well were studied by Abouelmagd and El-Shaboury [2]. They found that the collinear points are still unstable while the triangular points are stable for $0 < \mu < \mu_c$, and unstable for $\mu_c \le \mu \le 1/2$, where $\mu_c \in (0, 1/2)$. They also deduced that for these points the range of stability will decrease. In addition they studied the periodic orbits around these points in the range $0 < \mu < \mu_c$.

From a physical point of view, it is unreasonable to consider all objects as being point masses with no physical dimensions. This is conflict with the real cases for the celestial bodies. On the other hand the effect of rotation causes deformation in the shape of the objects at the poles as might be expected. For this reason most objects may be treated to a good approximation as oblate spheroids, in this context there are a number of contributions established by: Abouelmagd [5, 6], Abouelmagd et al. [7, 8], and Abouelmagd et al. [9–11]. Therefore in this paper we will generalize the restricted three-body, in which the three participating bodies are oblate spheroids as well as the existence of small perturbations in the Coriolis and centrifugal forces. The existence of libration points and their linear stability and the periodic orbits around these points will be studied. This model could be applicable in astrodynamics or astrophysics.

Now we assume that the three participating bodies are oblate spheroids. Therefore we refer to oblateness parameters of the bigger, smaller primaries and infinitesimal body by A_1, A_2, and A, respectively. We also denote the small perturbations in Coriolis and centrifugal forces by $\varepsilon_i : i = 1, 2$, where $0 < A_1 \ll 1, 0 < A_2 \ll 1$, $0 < A \ll 1$, and $\varepsilon_i \ll 1$.

2 The equations of motion

It is well known that there are five exact solutions in three-body problem which the three masses maintain a constant configuration which revolves with constant angular velocity. An important specialization of the three-body problem is the restricted three-body problem in which m is infinitesimal body, and m_1 and m_2 are the bigger and smaller primaries, respectively. They move in circular orbits around their barycenter. The smallness of m means that it does not influence the motion of m_1 and m_2. For many purposes, it is convenient to describe the motion of m in a coordinate system which is attached to both m_1 and m_2. In this rotating coordinate system, the five Lagrangian solutions show up as five fixed points at which m would be stationary if placed there with zero velocity.

If the object has axial symmetry, it can be shown from potential theory that the gravitational potential experienced by the satellite can be written as in (Murray and Dermott, [22]), namely,

$$V = -\frac{Gm_0}{r} \left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{R_0}{r} \right)^n p_n(\sin \delta) \right], \tag{1}$$

where

- G is the universal constant.

- m_0 and R_0 are the mass and the mean radius of the object.

- $p_n(\sin \delta)$ denote the Legendre polynomials of degree n.

- r is the distance from the center of the object to the satellite.

- J_n is the dimensionless coefficient that characterizes the size of non-spherical components of the potential.

If the motion of the satellite and the object are in the same plane ($\delta = 0$), then Eq. (1) becomes

$$V = -\frac{Gm_0}{r}\left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{R_0}{r}\right)^n p_n(0)\right], \tag{2}$$

where $p_{2n}(0) = \dfrac{(-1)^n(2n)!}{2^{2n}(n!)^2}, p_{2n+1}(0) = 0$. In addition, let us consider the first zonal harmonic. Hence, Eq. (6) can be reduced to

$$V = -\frac{Gm_0}{r}\left[\frac{1}{r} + \frac{J_2 R_0^2}{2r^3}\right]. \tag{3}$$

Now we assume that m_1, m_2 and m are the masses of the bigger, smaller and infinitesimal bodies, r_1 and r_2 are the distances from m_1 and m_2 to m, respectively.

Let m_1 and m_2 have a circular orbit about their common center of mass and m is moving under their gravitational field in the same plane but it is so small that it does not affect their motion. Furthermore let:

- $\underline{R} = ui + vj$ be the position vector of m_2 with respect to m_1.

- R is the separation distance between m_1 and m_2.

Using Eq. (3), then the potential V_{12} between m_1 and m_2 can be written as

$$V_{12} = -Gm_1 m_2 \left[\frac{1}{R} + \frac{A_1 + A_2}{2R^3}\right]. \tag{4}$$

Thus, the equation of motion for m_2 is

$$\ddot{\underline{R}} = -\left(\frac{m_1 + m_2}{m_1 m_2}\right)\nabla V_{12}, \tag{5}$$

where

$$\nabla = i\frac{\partial}{\partial u} + j\frac{\partial}{\partial v}.$$

Hence Eq. (5) has the following particular solutions:

(i) $R = $ constant.

(ii) $u = R\cos\theta$.

(iii) $v = R\sin\theta$.

(iv) $\theta = nt$.

Therefore, the perturbed mean motion is given by the following relation

$$n^2 = G(m_1 + m_2)\left[\frac{1}{R^3} + \frac{3(A_1 + A_2)}{2R^5}\right]. \tag{6}$$

Now we assume that XY is the sidereal frame while xy are the synodic frames such that they have the same origin at the center of mass for the primaries. We also chose the synodic frame to rotate with angular velocity n in positive direction (see Fig. (1)). Let the coordinates of m_1, m_2, and m in the sidereal frame be $(X_1, Y_1), (X_2, Y_2)$, and (X, Y), respectively, but in the synodic frame are $(x_1, y_1), (x_2, y_2)$, and (x, y), respectively, too.
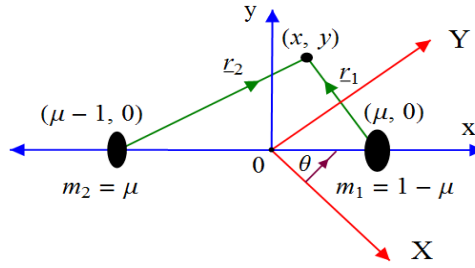
Fig. 1 Configuration of the sidereal *XY* and synodic *xy* frames of the restricted three-body problem.

Based on the above discussion, the sidereal coordinates are related to the synodic coordinates via the following relation:

$$X = x\cos nt - y\sin nt. \tag{7}$$

$$Y = x\sin nt + y\cos nt. \tag{8}$$

Using Newton's law, the equations of motion for infinitesimal body in a sidereal frame are as follows:

$$m\ddot{X} = \frac{\partial V}{\partial X}. \tag{9}$$

$$m\ddot{Y} = \frac{\partial V}{\partial Y}. \tag{10}$$

From Eq. (3), the potential V_1 between m and m_1 is given by

$$V_1 = -Gmm_1\left[\frac{1}{r_1} + \frac{A_1 + A}{2r_1^3}\right], \tag{11}$$

and the potential V between m, m_1 and m_2 is also given by

$$V_2 = -Gmm_2\left[\frac{1}{r_2} + \frac{A_2 + A}{2r_2^3}\right], \tag{12}$$

while the total potential V between m, m_1, and m_2 is controlled by

$$V = -Gmm_1\left[\frac{1}{r_1} + \frac{A_1 + A}{2r_1^3}\right] - Gmm_2\left[\frac{1}{r_2} + \frac{A_2 + A}{2r_2^3}\right], \tag{13}$$

once the notation and terminology of Szebehely [30] has been adopted. As a consequence, the distance between the primaries does not change and is equal to one, the sum of masses of the primaries is also taken as one, the unit time is chosen as to make the unperturbed mean motion and the gravitational constant unity, then it follows that $m_1 = 1 - \mu$, whose coordinate is $(\mu, 0), m_2 = \mu \leq 1/2$ and located at $(\mu - 1, 0)$.

Substituting Eqs. (7-8) and (11-13) into (9-138) that gives the equations of motion in a synodic coordinate system $(x - y)$ for infinitesimal body with dimensionless variables as

(i) $\ddot{x} - 2n\dot{y} = U_x.$

(ii) $\ddot{y} + 2n\dot{x} = U_y,$

where

$$U = \frac{1}{2}n^2\left[(1-\mu)r_1^2 + \mu r_2^2\right] + (1-\mu)\left[\frac{1}{r_1} + \frac{A_1 + A}{2r_1^3}\right] + \mu\left[\frac{1}{r_2} + \frac{A_2 + A}{2r_2^3}\right]. \tag{14}$$

$$n^2 = 1 + \frac{3}{2}(A_1 + A_2). \tag{15}$$

$$r_1^2 = (x - \mu)^2 + y^2. \tag{16}$$

$$r_2^2 = (x - \mu + 1)^2 + y^2. \tag{17}$$

Eqs. (i)-(ii) and (14) represent the equation of motion for the infinitesimal body in a rotating coordinate system when the three participating bodies are oblate spheroids.

In the previous equations the perturbed effects of Coriolis and centrifugal forces are switched off. In physics the Coriolis effect is a deflection of moving objects when they are viewed in a rotating reference frame. In a reference frame with clockwise rotation the deflection is to the left of the motion of the object, in one with counter-clockwise rotation the deflection is to the right. The Newton's laws of motion govern the motion of an object in a non-accelerating inertial frame of reference. When Newton's laws are transformed to a uniformly rotating frame of reference, the Coriolis and centrifugal forces appear.

Both forces are proportional to the mass of the object. The Coriolis force is proportional to the rotation rate and the centrifugal force is proportional to its square. The Coriolis force acts in a direction perpendicular to the rotation axis and to the velocity of the body in the rotating frame and is proportional to the object's speed in the rotating frame. The centrifugal force acts outwards in the radial direction and is proportional to the distance of the body from the axis of the rotating frame.

It is important to note that these forces do not arise from any physical agency: they arise solely as a result of rotation for the coordinate system. Thus, if the angular velocity is reduced to zero, both the Coriolis and the centrifugal forces will vanish. For this reason, the Coriolis and centrifugal forces are referred to as "fictitious" or "inertial" forces. These additional forces allow the application of Newton's laws to a rotating system. They are correcting factors that do not exist in a non-accelerating or inertial reference frame.

Anyway, if the perturbed effects of Coriolis and centrifugal forces are considered, let ϕ, ψ denote these forces, respectively. Assume that ε_1 represents the perturbation in Coriolis force, while ε_2 is the accompanying perturbation in the centrifugal force such that

$$\varphi = 1 + \varepsilon_1, \qquad \varepsilon_1 \ll 1. \tag{18}$$

$$\psi = 1 + \varepsilon_2, \qquad \varepsilon_2 \ll 1. \tag{19}$$
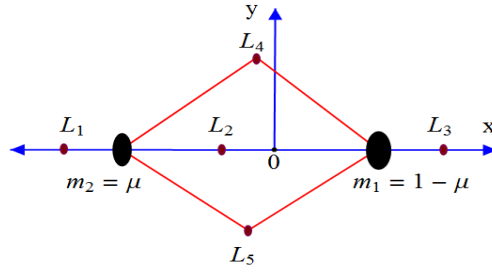
Hence, Eqs. (i)-(ii) and (14) can be written as follows:

$$\ddot{x} - 2\varphi n\dot{y} = \Omega_x. \tag{20}$$

$$\ddot{y} + 2\varphi n\dot{x} = \Omega_y, \tag{21}$$

where

$$\Omega = \frac{1}{2}n^2\psi\left[(1-\mu)r_1^2 + \mu r_2^2\right] + (1-\mu)\left[\frac{1}{r_1} + \frac{A_1 + A}{2r_1^3}\right] + \mu\left[\frac{1}{r_2} + \frac{A_2 + A}{2r_2^3}\right]. \tag{22}$$

It is worth mentioning that the subscripts on the right handside denote the partial derivation with respect to x and y, respectively.

Fig. 2 The positions of the equilibrium points $L_i : i = 1, \ldots, 5$.

3 The locations of the libration points

The libration points are singular points of the differential equations of motions in the restricted problem of three bodies. They are also equilibrium points since the gravitational force on a mass placed in such a point are balanced by the centrifugal force. The three libration points are called the collinear points which are found on the line connecting with the primaries. The other two are called the triangular points which are symmetrical with respect to this line and form triangles, see Fig. (1).

From Eqs. (i)-(91), we can obtain the Jacobi integral as

$$\dot{x}^2 + \dot{y}^2 - 2\Omega + c = 0, \tag{23}$$

where c is a constant of integration.

The locations of these points will be determined by solving the following expressions:

$$\Omega_x = \Omega_{\dot{x}} = \Omega_y = \Omega_{\dot{y}} = 0, \tag{24}$$

where the first partial derivatives of Ω will be governed by

$$\Omega_x = (x - \mu)f_1(r_1) + (x - \mu + 1)f_2(r_2). \tag{25}$$

$$\Omega_y = y[f_1(r_1) + f_2(r_2)]. \tag{26}$$

$$f_1(r_1) = (1 - \mu)\left[\psi n^2 - \left(\frac{1}{r_1^3} + \frac{3(A_1 + A)}{2r_1^5}\right)\right]. \tag{27}$$

$$f_2(r_2) = \mu\left[\psi n^2 - \left(\frac{1}{r_2^3} + \frac{3(A_2 + A)}{2r_2^5}\right)\right]. \tag{28}$$

From Eqs. (25-26), two cases yield. They will be investigated in the following subsection.

3.1 The collinear points

The collinear points $L_i : i = 1, 2, 3$ are the solutions of Eqs. (24) and (25-26) when $y = 0$. Therefore,

$$v = (-1)^i \frac{r_2^4(r_1^5 - pr_1^2 - a)}{r_1^4(r_2^5 - pr_2^2 - b)} : \qquad i = 1, 2, 3, \tag{29}$$

where i indicates the value of v at $L_i : i = 1, 2, 3$. Further,

$$v = \frac{\mu}{1 - \mu}, \quad p = \frac{1}{q}, \quad q = \psi n^2, \tag{30}$$

$$a = \frac{3(A_1 + A)}{2q}, \quad b = \frac{3(A_2 + A)}{2q}. \tag{31}$$

Lagrange developed a useful method for inverting series expansions which could be applicable in the present work. He investigated that if a variable χ can be expressed as a function of $\mathscr{Y}$ as in the below relation (Murray and Dermott, [22]):

$$\mathscr{Y} = \chi + \vartheta \phi(\Upsilon), \quad \vartheta < 1. \tag{32}$$

Therefore, Υ can be also expressed as a function of χ as follows:

$$\mathscr{Y} = \chi + \sum_{k=1}^{\infty} \frac{\vartheta^k}{k!} \frac{d^{k-1}}{d\chi^{k-1}} [\phi(\chi)]^k. \tag{33}$$

3.1.1   Location of L_1

The solution of L_1 lies beyond the smaller mass as in Fig. (2). Therefore, $r_1 - r_2 = 1$, and hence, $r_2 = \mu - x - 1, r_1 = \mu - x$, which implies that

$$\frac{\partial r_1}{\partial x} = \frac{\partial r_2}{\partial x} = -1,$$

provided that we take $r_2 = r, r_1 = 1 + r$, and substitute them into Eq. (29). After some simple calculations the value of v will be written as

$$v = a_4 r^4 + a_5 r^5 + a_6 r^6 + a_7 r^7 + \mathscr{O}[r^8], \tag{34}$$

where the values a_4, a_5, a_6 and a_7 are given as in Appendix I.

Now, we use Lagrange's inversion method to invert the above series and hence to express r as a function of v which is written as

$$r = \left(\frac{1}{a_4}\right)^{\frac{1}{4}} v^{\frac{1}{4}} - \frac{1}{4} \left(\frac{1}{a_4}\right)^{\frac{3}{2}} a_5 \sqrt{v} + \frac{1}{32} \left(\frac{1}{a_4}\right)^{\frac{11}{4}} (7a_5^2 - 8a_4 a_6) v^{\frac{3}{4}} + \frac{-a_5^3 + 2a_4 a_5 a_6 - a_4^2 a_7}{4a_4^4} v + \mathscr{O}[v^{\frac{5}{4}}]. \tag{35}$$

Consequently, the location of L_1 will be given by

$$x_1 = -1 + \frac{v}{1+v} - \left(\frac{1}{a_4}\right)^{\frac{1}{4}} v^{\frac{1}{4}} + \frac{1}{4} \left(\frac{1}{a_4}\right)^{\frac{3}{2}} a_5 \sqrt{v} - \frac{1}{32} \left(\frac{1}{a_4}\right)^{\frac{11}{4}} (7a_5^2 - 8a_4 a_6) v^{\frac{3}{4}} - \frac{-a_5^3 + 2a_4 a_5 a_6 - a_4^2 a_7}{4a_4^4} v. \tag{36}$$

3.1.2   Location of L_2

The solution of L_2 lies between the two finite bodies as in Fig. (2), where $r_1 + r_2 = 1$. Hence, $r_2 = x - \mu + 1, r_1 = \mu - x$. This implies that

$$\frac{\partial r_1}{\partial x} = -\frac{\partial r_2}{\partial x} = -1.$$

Thus, if we take $r_2 = \rho, r_1 = 1 - \rho$, and substitute them into Eq. (29), then the value of v will be given by

$$v = b_4 \rho^4 + b_5 \rho^5 + b_6 \rho^6 + b_7 \rho^7 + \mathscr{O}[\rho^8],$$

where the values $b_4, b_5, b_6,$ and b_7 are governed as in Appendix I.

As mentioned previously, we use Lagrange's inversion method in order to invert the above series and hence, express ρ as a function of v, which can be written as

$$\rho = \left(\frac{1}{b_4}\right)^{\frac{1}{4}} v^{\frac{1}{4}} - \frac{1}{4} \left(\frac{1}{b_4}\right)^{\frac{3}{2}} b_5 \sqrt{v} + \frac{1}{32} \left(\frac{1}{b_4}\right)^{\frac{11}{4}} (7b_5^2 - 8b_4 b_6) v^{\frac{3}{4}} + \frac{(-b_5^3 + 2b_4 b_5 b_6 - b_4^2 b_7)}{4b_4^4} v + \mathscr{O}[v^{\frac{5}{4}}]. \tag{37}$$

Therefore, the location of L_2 could be written as

$$x_2 = -1 + \frac{v}{1+v} + \left(\frac{1}{b_4}\right)^{\frac{1}{4}} v^{\frac{1}{4}} - \frac{1}{4} \left(\frac{1}{b_4}\right)^{\frac{3}{2}} b_5 \sqrt{v} + \frac{1}{32} \left(\frac{1}{b_4}\right)^{\frac{11}{4}} (7b_5^2 - 8b_4 b_6) v^{\frac{3}{4}} + \frac{(-b_5^3 + 2b_4 b_5 b_6 - b_4^2 b_7)}{4b_4^4} v. \tag{38}$$

3.1.3 Location of L_3

The solution of L_3 lies beyond the large mass as in Fig. (2), where $r_2 - r_1 = 1$. Hence, $r_2 = 1 - \mu + x, r_1 = x - \mu$. This implies that

$$\frac{\partial r_1}{\partial x} = \frac{\partial r_2}{\partial x} = 1.$$

Thus, if we take $r_1 = 1 - + \delta, r_2 = 2 + \delta$, and substitute them into Eq. (29), then v is as follows:

$$v = c_0 + c_1 \delta + c_2 \delta^2 + \mathcal{O}[\delta^3], \tag{39}$$

where c_0, c_1 and c_2 are given as in Appendix I.

We also use Lagrange's inversion method to invert the above series and to express δ as a function of v. Hence, the following expression yields:

$$\delta = \frac{v - c_0}{c_1} - \frac{c_2(v - c_0)^2}{c_1^3} + \mathcal{O}[(v - c_0)^3]. \tag{40}$$

Thus, the location of L_3 will be controlled by

$$x_3 = 1 + \frac{v}{1 + v} + \frac{v - c_0}{c_1} - \frac{c_2(v - c_0)^2}{c_1^3}. \tag{41}$$

Note that Eqs. (36), (38) and (41) represent semi-closed forms for the positions of collinear points. Furthermore, the small perturbation in the Coriolis force has no effect on these positions.

3.1.4 The triangular points

The solutions of Eqs. (24) and (25-26) when $y \neq 0$ will give the triangular points L_4 and L_5. These solutions will lead to

$$f_1(r_1) = 0 = f_2(r_2). \tag{42}$$

Therefore, we get

$$\psi n^2 = \left(\frac{1}{r_1^3} + \frac{3(A_1 + A)}{2r_1^5} \right). \tag{43}$$

$$\psi n^2 = \left(\frac{1}{r_2^3} + \frac{3(A_2 + A)}{2r_2^5} \right). \tag{44}$$

To study the effect of oblateness in the perturbed problem on the locations of the triangular points, we will assume that the three participating bodies are not oblate spheroids, namely, $A_1 = A_2 = A = 0$.

Hence Eqs. (43-44) will admit $r_i = \frac{1}{\sqrt[3]{\psi}}$. Consequently if the three bodies are oblate, then the solution of Eqs. (43-44) will be controlled by

$$r_i = \frac{1}{\sqrt[3]{\psi}} + \pi_i, \quad \pi_i = 1 : i = 1, 2, \tag{45}$$

where π_i is the additive displacement in r_i as a result of oblateness effects. Substituting Eq. (45) into (43-44) and restricting ourselves to the linear terms in π_1, π_2, A_1, A_2, and A, we obtain an appropriate approach to π_i as follows:

$$\pi_1 = -\frac{1}{2}(A_1 + A_2)\psi^{-\frac{1}{3}} + \frac{1}{2}(A_1 + A)\psi^{\frac{1}{3}}. \tag{46}$$

$$\pi_2 = -\frac{1}{2}(A_1 + A_2)\psi^{-\frac{1}{3}} + \frac{1}{2}(A_2 + A)\psi^{\frac{1}{3}}. \tag{47}$$

From Eqs. (16-17), the exact solutions of the triangular points could be written as

$$x = -\frac{1}{2}\left[1 - 2\mu + r_1^2 - r_2^2\right]. \tag{48}$$

$$y = \pm\left[\left(\frac{r_1^2 + r_2^2}{2}\right) - \left(\frac{r_1^2 - r_2^2}{2}\right)^2 - \frac{1}{4}\right]^{\frac{1}{2}}. \tag{49}$$

Substituting Eqs. (46-47) and (45) into both (48-49) and limiting ourselves to the previous assumptions, it holds that

$$x = \mu - \frac{1}{2} - \frac{1}{2}(A_1 - A_2). \tag{50}$$

$$y = \pm\frac{\sqrt{3}}{2}\left[1 - \frac{1}{3}(A_1 + A_2 - 2A) - \frac{4}{9}(1 + A_1 + A_2 + A)\varepsilon_2\right]. \tag{51}$$

Eqs. (50-51) state that the presence of parameters that represent the oblateness of the infinitesimal body, the small perturbations in Coriolis and centrifugal forces have no effect on the $x-$coordinate of the triangular points. While the $y-$coordinate of the triangular points may be affected by all the parameters of the perturbed forces except for the small perturbations in the Coriolis force.

4 The stability of the libration points

The models which describe the restricted three-body problem are developed so far they provide a concise description regarding the third body motion. These models depend on the state variables that define the initial conditions and the variety of parameters that either affects the measurement process or dynamical motions, as well. There is a difficulty to solve these models directly for any choice of parameters from a given set of measurements, due to the great complexity underlying these models. This is the main reason for which our attention is focused on linearizing the dynamical system to obtain simplified expressions that could be handled more easily. To understand the possible motion of the third body in the vicinity of equilibrium points, the equations of motion should be linearized along the initial state vector. In other words, we will expand the right handside of motion equations around the equilibrium points. Consequently, the linear part in these equations is called the *variational equations*. We also assume that the variation vector is related to the initial state vector by the following expression:

$$\underline{r} = \underline{r}_0 + \Delta\underline{r}, \tag{52}$$

where $r \equiv (x,y), \underline{r}_0 \equiv (x_0, y_0)$, and $\Delta\underline{r} \equiv (\xi, \eta)$, provided that ξ and η are small enough. Thus, substituting Eq. (52) into Eq. (20-21), and expanding them via a Taylor series, one obtain that

$$\ddot{\xi} - 2n\varphi\dot{\eta} = \Omega_x^0 + \frac{1}{1!}\left[\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right]\Omega_x^0 + \frac{1}{2!}\left[\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right]^2\Omega_x^0 + \mathscr{O}[3]. \tag{53}$$

$$\ddot{\eta} + 2n\varphi\dot{\xi} = \Omega_y^0 + \frac{1}{1!}\left[\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right]\Omega_y^0 + \frac{1}{2!}\left[\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right]^2\Omega_y^0 + \mathscr{O}[3], \tag{54}$$

where $\mathscr{O}[3]$ stands from the third and higher order terms in ξ and η. When these terms and the second order terms are omitted, then Eqs. (53-54) give us the linear variational equations and ξ and η are called the *variations*. Indeed,

$$\ddot{\xi} - 2n\varphi\dot{\eta} = \xi\Omega_{xx}^0 + \eta\Omega_{xy}^0. \tag{55}$$

$$\ddot{\eta} + 2n\varphi\dot{\xi} = \xi\Omega_{xy}^0 + \eta\Omega_{yy}^0, \tag{56}$$

where the subscripts x and y denote the second partial derivatives of Ω and superscript 0 means that these derivatives are calculated with respect to one of the five equilibrium points (x_0, y_0).

Eqs. (55-56) represent the linear differential system of second order in $\mathbb{R}^2$. Hence, we transform such a system in a first order system in $\mathbb{R}^4$. To deal with, let us assume that $\xi_1 = \xi, \xi_2 = \eta, \xi_3 = \dot{\xi}_1, \xi_4 = \dot{\xi}_2$, and let also substitute them into Eqs. (55-56). Then the system can be written as follows:

$$\dot{\xi}_1 = \xi_3. \tag{57}$$

$$\dot{\xi}_2 = \xi_4. \tag{58}$$

$$\dot{\xi}_3 = \Omega_{xx}^0 \xi_1 + \Omega_{xy}^0 \xi_2 + 2n\varphi \xi_4. \tag{59}$$

$$\dot{\xi}_4 = \Omega_{xy}^0 \xi_1 + \Omega_{yy}^0 \xi_2 - 2n\varphi \xi_3. \tag{60}$$

The stability of libration points is a well-known problem explored alongn the classical literature. Therefore, in this section and also in forthcoming sections, we will follow the analysis based on Szebehely computations [29].

4.1 Characteristic equation

Let us consider a dynamical system within four degree of freedom which is described through the following set of differential equations:

$$\dot{\underline{\chi}} = \underline{g}(\underline{\chi}, t), \tag{61}$$

where $\underline{\chi} \in \mathbb{R}^4$, and $\underline{g} = (g_1, g_2, g_3, g_4)$ is a vector function. This system is non-autonomous whether the vector function $\underline{g}$ explicitly depends on time. Otherwise, it is autonomous and can be describes in the following form:

$$\dot{\underline{\chi}} = \underline{g}(\underline{\chi}), \tag{62}$$

Thus, the solution of the set of equations $\underline{g}(\underline{\chi}_0) = 0$ for the system at Eq. (62) gives us an equilibrium point $\underline{\chi} = \underline{\chi}_0$. Hence, the solution of the system in Eq. (62) at time t can be written as $\underline{\chi} = \underline{\chi}(\underline{\chi}_0, t) : t \in \mathbb{R}$, in terms of the initial state vector $\underline{\chi}_0$. Further, let $\underline{\chi} = (x_1, x_2, x_3, x_4)^S$, where S denotes the transposed vector. Hence, if M is the matrix of the dynamical system, then the linearized system can be written as

$$\dot{\underline{\chi}} = M\underline{\chi}. \tag{63}$$

It is worth mentioning that this system has a non-trivial solution if and only if

$$\det(\lambda I - M) = 0, \tag{64}$$

where λ is an eigenvalue of matrix M or a root of the characteristic polynomial appeared in Eq. (64). Moreover, if $\underline{e}_i$ is an eigenvector associated with λ_i, then the relation $M\underline{e}_i = \lambda \underline{e}_i$ will be achieved. Consequently, the general solution of Eq. (63) can be written as

$$\underline{\chi}(t) = \sum_{i=1}^{4} c_i e^{\lambda_i t} \underline{e}_i. \tag{65}$$

The properties of stability for $t \geq t_0$ of the linearized system depend on the nature regarding the eigenvalues of the matrix M. According to that, next we distinguish three main cases:

(i) If all the eigenvalues are negative real numbers, then the solution becomes stable. However, if any one of them is positive, then the solution is unstable.

(ii) If the eigenvalues are complex numbers having negative real parts, then the equilibrium point becomes asymptotically stable. Otherwise, if some of the eigenvalues or all of them have positive real parts, then the equilibrium point becomes unstable.

(iii) If the eigenvalues are pure imaginary numbers, then the solution is stable. Nevertheless, if there are multiple roots, then the solution contains periodic and secular terms, so the equilibrium point becomes unstable (see Szebehely [29] and Celletti (2010)).

From the previous discussion, we can write the system in Eq. (57)-(60) in the form

$$\dot{\mathscr{R}} = M\mathscr{R}, \tag{66}$$

where

$$\mathscr{R} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix}, \qquad \dot{\mathscr{R}} = \begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{\xi}_4 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \Omega_{xx}^0 & \Omega_{xy}^0 & 0 & 2n\varphi \\ \Omega_{xy}^0 & \Omega_{yy}^0 & -2n\varphi & 0 \end{pmatrix}.$$

Thus, M is the matrix of the linear system such that the derivatives of the potential functions are be governed by the following expressions:

$$\Omega_{xx} = f_1(r_1) + f_2(r_2) + \frac{1}{r_1}\frac{\partial f_1}{\partial r_1}(x-\mu)^2 + \frac{1}{r_2}\frac{\partial f_2}{\partial r_2}(x-\mu+1)^2. \tag{67}$$

$$\Omega_{xy} = \left[\frac{1}{r_1}\frac{\partial f_1}{\partial r_1}(x-\mu) + \frac{1}{r_2}\frac{\partial f_2}{\partial r_2}(x-\mu+1)\right]y. \tag{68}$$

$$\Omega_{yy} = f_1(r_1) + f_2(r_2) + \left[\frac{1}{r_1}\frac{\partial f_1}{\partial r_1} + \frac{1}{r_2}\frac{\partial f_2}{\partial r_2}\right]y^2, \tag{69}$$

where

$$\frac{\partial f_1}{\partial r_1} = (1-\mu)\left[\frac{3}{r_1^4} + \frac{15(A_1+A)}{2r_1^6}\right]. \tag{70}$$

$$\frac{\partial f_2}{\partial r_2} = \mu\left[\frac{3}{r_2^4} + \frac{15(A_2+A)}{2r_2^6}\right]. \tag{71}$$

Further, the characteristic equation corresponding to Eq. (39) is as follows:

$$\lambda^4 + (4n^2\varphi^2 - \Omega_{xx}^0 - \Omega_{yy}^0)\lambda^2 + \Omega_{xx}^0\Omega_{yy}^0 - (\Omega_{xy}^0)^2 = 0. \tag{72}$$

The most fundamental questions regarding the motion near libration points are those about the stability of such points. Thus, in the next subsections we will survey the results obtained regarding the stability.

4.2   Stability of collinear points

To investigate the stability of collinear points, we have to study the motion in the vicinity of such points. For this purpose, Eqs. (67-69) could be written at the collinear points as follows:

$$\Omega_{xx}^0 = \psi n^2 + (1-\mu)\left(\frac{2}{r_1^3} + \frac{6(A_1+A)}{r_1^5}\right) + \mu\left(\frac{2}{r_2^3} + \frac{6(A_2+A)}{r_2^5}\right). \tag{73}$$

$$\Omega_{xy}^0 = 0. \tag{74}$$

$$\Omega_{yy}^0 = \psi n^2 - (1-\mu)\left(\frac{1}{r_1^3} + \frac{3(A_1+A)}{2r_1^5}\right) - \mu\left(\frac{1}{r_2^3} + \frac{3(A_2+A)}{2r_2^5}\right). \tag{75}$$

It is worth mentioning that Eqs. (73)-(76) show that $\Omega_{xx}^0 > 0, \Omega_{xy}^0 = 0$, and $\Omega_{yy}^0 < 0$ at $L_i : i = 1,2,3$. Thus, as a result of this,

$$\Omega_{xx}^0\Omega_{yy}^0 - (\Omega_{xy}^0)^2 < 0. \tag{76}$$

Let $\lambda^2 = \Lambda$. Therefore, Eq. (72) could be written as

$$\Lambda^2 + \overline{\lambda}_{1c}\Lambda + \overline{\lambda}_{2c} = 0, \tag{77}$$

where

$$\overline{\lambda}_{1c} = 4n^2\varphi^2 - \Omega_{xx}^0 - \Omega_{yy}^0. \tag{78}$$

$$\overline{\lambda}_{2c} = \Omega_{xx}^0\Omega_{yy}^0. \tag{79}$$

Hence, the roots of Eq. (77) are

$$\Lambda_{1,2} = -\frac{1}{2}\left[\overline{\lambda}_{1c} \mp \sqrt{\overline{\lambda}_{1c}^2 - 4\overline{\lambda}_{2c}}\right]. \tag{80}$$

Eqs. (78-79) and (80) show that the roots of the characteristic Eqs. (72) could be written as $\lambda_{1,2} = \pm\sigma, \lambda_{3,4} = \pm i\tau$, where σ and τ are real numbers. Furthermore, it holds that

$$\sigma^2 = \frac{1}{2}\left[\sqrt{\overline{\lambda}_{1c}^2 - 4\overline{\lambda}_{2c}} - \overline{\lambda}_{1c}\right]. \tag{81}$$

$$\tau^2 = \frac{1}{2}\left[\sqrt{\overline{\lambda}_{1c}^2 - 4\overline{\lambda}_{2c}} + \overline{\lambda}_{1c}\right]. \tag{82}$$

In addition, the general solution of Eqs. (53-54) can be written as

$$\xi(t) = \sum_{i=1}^{4}\delta_i e^{\lambda_i t}, \qquad \eta(t) = \sum_{i=1}^{4}\rho_i e^{\lambda_i t}. \tag{83}$$

Therefore,

$$\dot{\xi}(t) = \sum_{i=1}^{4}\lambda_i\delta_i e^{\lambda_i t}, \qquad \dot{\eta}(t) = \sum_{i=1}^{4}\lambda_i\rho_i e^{\lambda_i t}. \tag{84}$$

Hence, we can establish that the motion around the collinear points is unbounded. However, it holds that $\lambda_{3,4}$ are pure imaginary, and $\lambda_{1,2}$ are real. Accordingly, the solutions of these points become unstable.

4.3 Stability of triangular points

In this case, Eqs. (67-69) could be written as follows:

$$\Omega_{xx}^0 = \frac{3}{4}\left[1 + \frac{5}{2}(A_1 + A_2) + 2(1 - 2\mu)(A_1 - A_2)\right] + \frac{5}{4}\left[1 + \frac{5}{2}(A_1 + A_2) + 2(1 - 2\mu)(A_1 - A_2)\right]\varepsilon_2. \tag{85}$$

$$\begin{aligned}\Omega_{xy}^0 = &\mp\frac{3\sqrt{3}}{4}\left[1 - 2\mu + \frac{19}{6}\left(1 - \frac{26}{19}\mu\right)A_1 + \frac{7}{6}\left(1 - \frac{26}{7}\mu\right)A_2 + \frac{2}{3}(1 - 2\mu)A\right] \\ &+ \frac{11}{9}\left[1 - 2\mu + \frac{80}{297}(1 - 14\mu)(A_1 + A_2) + \frac{560}{297}\left(1 + \frac{4}{7}\mu\right)A\right]\varepsilon_2.\end{aligned} \tag{86}$$

$$\Omega_{yy}^0 = \frac{9}{4}\left[1 + \frac{11}{6}(A_1 + A_2) + \frac{4}{3}A\right] + \frac{7}{4}\left[1 + \frac{27}{14}(A_1 + A_2) + \frac{36}{7}A\right]\varepsilon_2. \tag{87}$$

Let $\lambda^2 = \omega$. Then Eq. (72) leads to

$$\omega^2 + \overline{\lambda}_{1t}\omega + \overline{\lambda}_{2t} = 0, \tag{88}$$

where

$$\overline{\lambda}_{1t} = 4n^2\varphi^2 - \Omega_{xx}^0 - \Omega_{yy}^0. \tag{89}$$

$$\overline{\lambda}_{2t} = \Omega^0_{xx}\Omega^0_{yy} - (\Omega^0_{yy})^2, \tag{90}$$

where the superscript 0 refers to the second partial derivatives which have to be evaluated at the triangular points. Hence,

$$\omega_{1,2} = -\frac{1}{2}\left[\overline{\lambda}_{1t} \mp \sqrt{D}\right], \tag{91}$$

where $D = \overline{\lambda}_{1t}^2 - 4\overline{\lambda}_{2t}$ is the discriminant of the quadratic appeared in Eq. (ii). Thus, we can write the discriminant as

$$D = \alpha\mu^2 + \beta\mu + \gamma, \tag{92}$$

where α, β, and γ have to be evaluated as in Appendix 2.

The variations ξ and η will represent stable solutions in the proximity of L_4 and L_5, provided that the four roots of the characteristic Eq. (72) are purely imaginary numbers. Otherwise, if any of such roots are real or complex, then these solutions will increase with time, and therefore, the solutions become unstable. Moroever, from Eq. (92), it is easy to show that

$$D_{\mu=0} > 0, \quad D_{\mu=1/2} < 0, \quad \frac{dD}{d\mu} \le 0, \quad \text{for all } \mu \in (0, 1/2). \tag{93}$$

Eq. (93) establishes that D is a decreasing function on the interval $(0, 1/2)$. Consequently, there is only one value of μ, say μ_c, on that interval for which D equals zero and D is positive provided that $0 < \mu < \mu_c$. In that case, the solutions are stable.

4.4   Critical mass

From the discussion above, it holds that the value of μ for which D equals zero is called the *critical mass value* (see Eq. (92)). This value will be governed by

$$\mu_c = -\frac{1}{2\alpha}\left[\beta + \sqrt{\beta^2 - 4\alpha\gamma}\right]. \tag{94}$$

Thus, if we substitute α, β, and γ into Eq. (94) and restrict ourselves to the linear terms of A_i, A, ε_i, as well as to the coupling terms in $A_i\varepsilon_i$ and $A\varepsilon_i : i = 1, 2$, then the critical mass value could be written as

$$\mu_c = \mu_{00} + p, \tag{95}$$

where $\mu_{00} = (9 - \sqrt{69})/18$ is the critical value given by Szebehely [30], when the three participating bodies are considered as point masses and the effects of small perturbations of Coriolis and centrifugal forces are neglected. Further, p represents these effects when the three participating bodies are oblate spheroids. Accordingly,

$$p = \mu_{10} - \mu_{01}A_1 + \mu_{02}A_2 - \mu_{03}A - (\mu_{11}\varepsilon_1 - \mu_{21}\varepsilon_2)A_1 + (\mu_{12}\varepsilon_1 - \mu_{22}\varepsilon_2)A_2 - (\mu_{13}\varepsilon_1 - \mu_{23}\varepsilon_2)A, \tag{96}$$

where $\mu_{00}, \mu_{10}, \mu_{i1}, \mu_{i2}$, and μ_{i3} are as in Appendix 3.

It is worth mentioning that Eq. (96) consists of three main parts. The first part gives the effects of small perturbations ε_1 and ε_2 in Coriolis and centrifugal forces, respectively. The second part represents oblateness influence of the three participating bodies, namely, A_1, A_2, and A_3. Finally, the third part represents the mixed effects for only one of the parameters A_1, A_2, A, together with ε_1 and ε_2. Thus, that part will disappear provided that one of the parameters is ignored.

5 Periodic orbits

The periodic orbits have great significance generally in the celestial mechanics and specially in space dynamics. That significance makes necessity for studying the periodic orbits. There are some important reasons

that motivated this study. First of all, note that the non-integrable dynamical systems prevent us to obtain a complete solution for any orbits unless they are asymptotic or periodic orbits. For any particular solution of the restricted three-body problem, one can always find out a periodic solution. We can also get periodic orbits via linearized solutions for the motion of infinitesimal body around the libration points. Periodic orbits play a crucial role in order to classify the different classes of orbits and also to reduce the dimensionality of the problem in the phase space. Additionally, they can be used as reference orbits.

5.1 Periodic orbits around collinear points

It is easy to find out periodic orbits about the collinear points $L_{1,2,3}$. However, these points are unstable, i.e., if a body in any of these points is disturbed, then it will move away. This is quite reasonable from a physical point of view. In fact, for these points there are saddle points in the potential function. Thus, substituting Eq. (83) into (55-56), simple calculations give the relations between the coefficients ρ_i and δ_i. In fact, it holds that

$$\rho_i = n_i \delta_i, \tag{97}$$

where

$$n_i = \pm \sqrt{\frac{\lambda_i^2 - \Omega_{xx}^0}{\Omega_{yy}^0 - \lambda_i^2}}. \tag{98}$$

These relations show that the coefficients δ_i and $\rho_i : i = 1, 2, 3, 4$ are dependent. Therefore, the four initial conditions $\xi_0, \eta_0, \dot{\xi}_0$ and $\dot{\eta}_0$ associated with Eqs. (55-56) will determine the two sets of coefficients. In particular, each set includes eight constants δ_i and ρ_i, where the subscript 0 refers to these quantities have to be evaluated at the initial time ($t = t_0$). Hence, substituting Eq. (97) into both Eqs. (83) and (84), the following expressions hold:

$$\xi_0 = \sum_{i=1}^{4} \delta_i e^{\lambda_i t}. \tag{99}$$

$$\eta_0 = \sum_{i=1}^{4} n_i \delta_i e^{\lambda_i t}. \tag{100}$$

$$\dot{\xi}_0 = \sum_{i=1}^{4} \lambda_i \delta_i e^{\lambda_i t}. \tag{101}$$

$$\dot{\eta}_0 = \sum_{i=1}^{4} n_i \lambda_i \delta_i e^{\lambda_i t}. \tag{102}$$

Since the determinant (Δ) of the system appeared in Eqs. (99)-(102) is not zero, then

$$\Delta = -\sqrt{\frac{\Omega_{xx}^0}{\Omega_{yy}^0}} (\sigma^2 + \tau^2)^2 \neq 0. \tag{103}$$

The coefficients in that system can be written as functions of the initial conditions. Thus, if the initial conditions ξ_0 and η_0 are properly chosen for $\delta_1 = \delta_2 = 0$, then Eq. (83 could be written in the form

$$\xi = \xi_0 \cos \tau(t - t_0) + \frac{\eta_0}{m_3} \sin \tau(t - t_0). \tag{104}$$

$$\eta = \eta_0 \cos \tau(t - t_0) - \xi_0 m_3 \sin \tau(t - t_0). \tag{105}$$

$$n_3 = i m_3. \tag{106}$$

ÜP⁴

Thus, from Eqs. (104-106), the following holds:

$$\dot{\xi}_0 = \frac{\eta_0 \tau}{m_3}.$$

(107)

$$\dot{\eta}_0 = -\xi_0 m_3 \tau.$$

(108)

This means that once the components of the initial conditions ξ_0 and η_0 are chosen, then we cannot select the associated initial velocities $\dot{\xi}_0$ and $\dot{\eta}_0$ as wished. Hence, after solving Eqs. (104-106) together in order to eliminate the time component, then the periodic orbits could be written in the form

$$\frac{\xi^2}{(\eta_0^2 + \xi_0^2 m_3^2)/m_3^2} + \frac{\eta^2}{\eta_0^2 + \xi_0^2 m_3^2} = 1.$$

(109)

Eq. (109) establishes that the trajectory of the infinitesimal body around the collinear points becomes an ellipse whose center is located at these points. Further, the semi-major (a_c) and the semi-minor b_c axes are parallel to the $y-$axis and the $x-$axis, respectively. Accordingly, these axes, the eccentricity (e_c), and the period (T_c) can be written in the following form:

$$a_c^2 = \eta_0^2 + \xi_0^2 m_3^2.$$

(110)

$$b_c^2 = (\eta_0^2 + \xi_0^2 m_3^2)/m_3^2.$$

(111)

$$e_c^2 = 1 - \frac{1}{m_3^2}.$$

(112)

$$T_c = 2\pi/\tau.$$

(113)

Since $\dot{\eta}_0 = -\xi_0 m_3 \tau, \dot{\xi}_0 = 0$ at $\xi_0 \neq 0$, and $\eta_0 = 0$, then we conclude that the motion along the orbits is retrograde.

5.2 Periodic orbits around triangular points

Since the triangular points are linearly stable provided that $0 < \mu < \mu_c$, and the characteristic equation presents four purely imaginary roots, then we have a bounded motion around the triangular points which consists of two harmonic motions. Consequently, this motion will be governed by

$$\xi = C_1 \cos s_1 t + D_1 \sin s_1 t + C_2 \cos s_2 t + D_2 \sin s_2 t.$$

(114)

$$\eta = \overline{C}_1 \cos s_1 t + \overline{D}_1 \sin s_1 t + \overline{C}_2 \cos s_2 t + \overline{D}_2 \sin s_2 t,$$

(115)

where s_1 and s_2 are the angular frequencies with respect to long and short periodic orbits, respectively. Moreover, the terms within the coefficients $C_1, D_1, \overline{C}_1$, and $\overline{D}_1$, are the long periodic terms, while the coefficients $C_2, D_2, \overline{C}_2$, and $\overline{D}_2$ are the short periodic terms. In addition, $s_{1,2}^2 = -\omega_{1,2}$. Hence, after substituting Eq. (92) into Eq. (91) and simplifying, it holds that the angular frequencies s_1 and s_2 will be given by

$$s_1 = \frac{3\sqrt{3}}{2}\sqrt{\mu}\left(1 - \frac{1}{2}\mu\right)\left(1 - \frac{4}{3}\varepsilon_1 + \frac{53}{18}\varepsilon_2 + \frac{1}{6}\left(33 - \frac{52}{3}\varepsilon + \frac{653}{18}\varepsilon_2\right)(A_1 + A_2) + \frac{1}{3}\left(11 - \frac{4}{3}\varepsilon_1 + \frac{65}{18}\varepsilon_2\right)A\right)$$

(116)

$$\begin{aligned}s_2 = {}&1 + 8\varepsilon_1 - 3\varepsilon_2 - \frac{27}{8}\left(1 - \frac{8}{3}\varepsilon_1 + \frac{49}{9}\varepsilon_2\right)\mu(1 - \mu) + \frac{3}{4}(1 + 8\varepsilon_1 - 3\varepsilon_2)(A_1 + A_2)\\ &- \frac{3}{2}\left(1 + \frac{5}{3}\varepsilon_2\right)[(A_1 + A) - \mu(A_1 - A_2)]\\ &- \frac{9}{8}\mu(1 - \mu)\left[\left(13 - 104\varepsilon_1 + \frac{215}{3}\varepsilon_2\right)(A_1 + A_2) + 4\left(1 - 8\varepsilon_1 + \frac{19}{3}\varepsilon\right)A\right]\end{aligned}$$

(117)

Eqs. (116-117) give the frequencies for the orbits of the long and short periodic motion when the influence of oblateness, small perturbations of Coriolis (ε_1), and centrifugal (ε_2) forces are considered. We would like also to show that these equations are valid in the range $0 < \mu < \mu_c$, where μ_c is the critical mass value.

Substituting Eqs. (114-115) into Eqs. (55-56), and equaling the coefficients of sine and cosine terms, respectively, then the following relation among the coefficients of the long and short periodic terms yields:

$$\overline{C}_i = \Gamma_i \left[2\varphi n s_i D_i - \Omega_{xy}^0 C_i\right]: \quad i = 1, 2. \tag{118}$$

$$\overline{D}_i = -\Gamma_i \left[2\varphi n s_i C_i + \Omega_{xy}^0 D_i\right]: \quad i = 1, 2, \tag{119}$$

where

$$\Gamma_i = \frac{s_i^2 + \Omega_{xx}^0}{4\varphi^2 n^2 s_i^2 + (\Omega_{xy}^0)^2} = \frac{1}{s_i^2 + \Omega_{yy}^0}: \quad i = 1, 2, \tag{120}$$

and $\Omega_{xx}^0, \Omega_{xy}^0$, and Ω_{yy}^0 are given as in Eqs. (85-87). We can eliminate the short periodic or the long periodic terms in the solution provided that the initial conditions are properly chosen. Hence, we can assume that $\{C_2 = D_2 = \overline{C}_2 = \overline{D}_2 = 0\}$ and $\{\xi_0, \eta_0, \dot{\xi}_0, \dot{\eta}_0\}$ are the initial conditions at $t = 0$, i.e., the short periodic terms are removed. Thus, substituting these quantities into Eqs. (114-115) and (118-119), then the following expressions hold:

$$C_1 = \xi_0. \tag{121}$$

$$\overline{C}_1 = \eta_0. \tag{122}$$

$$D_1 = \frac{\Omega_{xy}^0 \xi_0 + \eta_0 \left(s_1^2 + \Omega_{yy}^0\right)}{2\varphi n s_1}. \tag{123}$$

$$\overline{D}_1 = -\frac{\xi_0 \left(s_1^2 + \Omega_{xx}^0\right) + \eta_0 \Omega_{xy}^0}{2\varphi n s_1}. \tag{124}$$

$$\dot{\xi}_0 = \frac{\Omega_{xy}^0 \xi_0 + \eta_0 \left(s_1^2 + \Omega_{yy}^0\right)}{2\varphi n}. \tag{125}$$

$$\dot{\eta}_0 = -\frac{\xi_0 \left(s_1^2 + \Omega_{xx}^0\right) + \eta_0 \Omega_{xy}^0}{2\varphi n}. \tag{126}$$

Let us assume that the triangular points represent the origin of the coordinate system and also that the infinitesimal body starts its motion at the origin. Thus, from Eqs. (50-51) and (52), it holds that the initial conditions will be controlled by $(\xi_0, \eta_0) = (-x_0, -y_0)$, where

$$\xi_0 = -\mu + \frac{1}{2} + \frac{1}{2}(A_1 - A_2). \tag{127}$$

$$\eta_0 = \mp \frac{\sqrt{3}}{2}\left[1 - \frac{1}{3}(A_1 + A_2 - 2A) - \frac{4}{9}(1 + A_1 + A_2 + A)\varepsilon_2\right], \tag{128}$$

where the $\mp$ sign means that the origin is at L_4 (L_5).

5.3 Elliptic orbits

After we have removed the long or the short periodic terms, it holds that the path of the infinitesimal body will be an ellipse. We can justify this fact by rewriting Eqs. (114-115) with respect to the long periodic terms in the following form:

$$\xi = C_1 \cos s_1 t + D_1 \sin s_1 t. \tag{129}$$

$$\eta = \overline{C}_1 \cos s_1 t + \overline{D}_1 \sin s_1 t. \tag{130}$$

On the other hand, if we replace Eqs. (121-124) into Eqs. (129-130), and remove the time effect, then these expressions could be reduced to

$$\alpha_1 \xi^2 + 2\beta_1 \xi \eta + \eta^2 = \gamma_1. \tag{131}$$

This equation represents an ellipse centred at the origin of the coordinate system ξ and η, since

$$\begin{vmatrix} \alpha_1 & \beta_1 \\ \beta_1 & 1 \end{vmatrix} = (2\varphi n s_1 \Gamma_1)^2 > 0, \tag{132}$$

where

$$\alpha_1 = \frac{s_1^2 + \Omega_{xx}^0}{s_1^2 + \Omega_{yy}^0}. \tag{133}$$

$$\beta_1 = \frac{\Omega_{xy}^0}{s_1^2 + \Omega_{yy}^0}. \tag{134}$$

$$\gamma_1 = \alpha_1 \xi_0^2 + 2\beta_1 \xi_0 \eta_0 + \eta_0^2. \tag{135}$$

5.4   The orientation of principal axes of the ellipse

Since Eq. (131) includes a bilinear term of the form $\xi \eta$, then the principal axes of the ellipse are rotated by an angle θ corresponding to the coordinate system (ξ, η). This motivates the introduction of a new coordinate $(\overline{\xi}, \overline{\eta})$ such that the quadratic term only appears without the bilinear term. Thus, both the old and the new coordinate systems are governed by the following equations:

$$\xi = \overline{\xi} \cos \theta - \overline{\eta} \sin \theta. \tag{136}$$

$$\eta = \overline{\xi} \sin \theta + \overline{\eta} \cos \theta. \tag{137}$$

Hence, let us substitute Eqs. (136-137) into Eq. (131), and equate the coefficient of $\overline{\xi}\,\overline{\eta}$ to zero. Thus, the orientation of the principal axes is given as

$$\begin{aligned} \tan 2\theta = &\pm \sqrt{3} \left[1 - 2\mu + \frac{8}{3} \left(1 - \frac{19}{8}\mu \right) A_1 - \frac{4}{3} \left(1 + \frac{7}{4}\mu \right) A_2 - \frac{4}{3} (1+\mu) A \right] \\ &\pm \sqrt{3} \frac{8}{9} \left[\left[1 - 2\mu + \frac{13}{3} \left(1 - \frac{55}{26}\mu \right) A_1 - \frac{8}{3} \left(1 - \frac{5}{16}\mu \right) A_2 - \frac{11}{3} \left(1 + \frac{5}{11}\mu \right) A \right] \varepsilon_2 \right], \end{aligned} \tag{138}$$

where the $+$ sign refers to the center of the ellipse at L_4, while the $-$ sign corresponds to the L_5 case. Furthermore, it holds that the lengths of the semi-major (a_t) and the semi-minor (b_t) axes, the eccentricity (e_t), as well as the period of the motion (T_t), are controlled by

$$a_t^2 = \frac{2\gamma_1}{(1+\alpha_1) - (1-\alpha_1)\cos 2\theta + 2\beta_1 \sin 2\theta}. \tag{139}$$

$$b_t^2 = \frac{2\gamma_1}{(1+\alpha_1) + (1-\alpha_1)\cos 2\theta - 2\beta_1 \sin 2\theta}. \tag{140}$$

$$e_t^2 = \frac{2\left[(1-\alpha_1)\cos 2\theta - 2\beta_1 \sin 2\theta \right]}{(1+\alpha_1) + (1-\alpha_1)\cos 2\theta - 2\beta_1 \sin 2\theta}. \tag{141}$$

$$T_t^2 = \frac{2\pi}{s_i} : i = 1, 2. \tag{142}$$

Next, we will introduce an algorithm which allows to find out the elements in the periodic orbits around the equilibrium points. This algorithm will be formulated via the following steps:

(i) Determine the parameters $\mu, A_1, A_2, A, \varepsilon_1$, and ε_2, for any given system.

(ii) Evaluate $\Omega_{xx}^0, \Omega_{xy}^0$, and Ω_{yy}^0.

(iii) Evaluate τ.

(iv) Evaluate s_1 and s_2.

(v) Evaluate m_3.

(vi) Evaluate ξ_0 and η_0.

(vii) Evaluate α_1, β_1, and γ_1.

(viii) Evaluate $\sin 2\theta$ and $\cos 2\theta$.

(ix) Apply steps (v) and (vi) to obtain the lengths of the semi-major (a_c) and the semi-minor (b_c) axes as well as the value of the eccentricity (e_c).

(x) Apply step (v) to find the period of motion T_c.

(xi) Use steps (vii) and (viii) to get the lengths of the semi-major (a_t) and the semi-minor (b_t) axes as well as the value of the eccentricity (e_t).

(xii) Use step (iv) to find the period of motion T_t.

(xiii) Use steps (ix) and (x) for the periodic orbits around the collinear points.

(xiv) Use steps (xi) and (xii) for the periodic orbits around the triangular points.

6 Conclusions

The restricted three-body problem is investigated in the framework the three participating are oblate spheroids, as well as the effect of small perturbations in the Coriolis and the centrifugal forces. The existence of libration points and their linear stability have been studied, under the parameter effects consisting of the oblateness and the small perturbations in the Coriolis and the centrifugal forces. The Lagrangian method is applied to invert expansion series in order to determined the locations of the collinear points by a semi-closed formula. We state that under the aforementioned effects, the collinear points remain unstable and the triangular points are stable for $0 < \mu < \mu_c$, and unstable for $\mu_c \leq \mu \leq 1/2$.

We prove that the trajectories of the infinitesimal body around the five libration points are ellipses. In this context the elements of these ellipses (the frequencies, the semi-major,the semi-minor axes, the eccentricity and the period of motion) are also found. In addition, the directions of the principal axes for the orbits around the triangular points and the coefficients of long and short periodic terms have been evaluated. Furthermore, an algorithm has been provided in order to calculate the elements of the periodic orbits around such points.

Appendix I: The series of coefficients at the collinear points

• The series of coefficients at L_1 in Eqs. (20-21) are given as follows:

$$a_4 = \frac{1-a-p}{b}, \qquad a_5 = \frac{1+4a+2p}{b}.$$

$$a_6 = -\frac{10ab+p-ap+3bp+p^2}{b^2}, \qquad a_7 = \frac{20ab-p-4ap+4bp-2p^2}{b^2}.$$

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- The series of coefficients at L_2 in Eq. (24) are given by:

$$b_4 = \frac{a+p-1}{b}, \qquad b_5 = \frac{1+4a+2p}{b}.$$

$$b_6 = \frac{10ab+p-ap+3bp+p^2}{b^2}, \qquad b_7 = \frac{20ab-p-4ap+4bp-2p^2}{b^2}.$$

- The series of coefficients at L_3 in Eq. (29) are as follows:

$$c_0 = \frac{16(a+p-1)}{32-b-4p}, \qquad c_1 = -\frac{16\left(16+144a-3b-2ab-12ap+72p-4p^2\right)}{(32-b-4p)^2}.$$

$$c_2 = \frac{8}{(32-b-4p)^3}(512+25088a-192b-608ab-7b^2+7ab^2+8384p-4160ap+66bp$$
$$+70abp-40p^2+200ap^2+b^2p-832p^2+6bp^2+40p^3).$$

Appendix II: The coefficients of the discriminant at Eq. (63)

$$\alpha = 27\left[1+\frac{13}{3}(A_1+A_2)+\frac{4}{3}A+\frac{22}{9}\left[1+\frac{49}{11}(A_1+A_2)+\frac{20}{11}A\right]\varepsilon_2\right].$$

$$\beta = -27\left[1+\frac{1}{9}(37A_1+41A_2)+\frac{4}{3}A+\frac{2}{9}[11\varepsilon_2-(16\varepsilon_1-160\varepsilon_2)A_1+(16\varepsilon_1+134\varepsilon_2)A_2+20\varepsilon_2A]\right].$$

$$\gamma = 1-3(A_1-A_2+2A)-10\varepsilon_1A_1+(16\varepsilon_1-6\varepsilon_2)(1+3A_2)-8(6\varepsilon_1-\varepsilon_2)A.$$

Appendix III: The coefficients of the oblateness parameters for the critical mass in Eqs. (67-69)

$$\mu_{00} = \frac{1}{2}\left(1-\frac{\sqrt{69}}{9}\right), \qquad \mu_{10} = \frac{4}{27\sqrt{69}}(36\varepsilon_1-19\varepsilon_2).$$

$$\mu_{01} = \frac{1}{9}\left(1+\frac{13}{\sqrt{69}}\right), \qquad \mu_{11} = \frac{16}{9}\left(1+\frac{325}{23\sqrt{69}}\right).$$

$$\mu_{21} = \frac{61}{81}\left(1+\frac{20091}{1403\sqrt{69}}\right).$$

$$\mu_{02} = \frac{1}{9}\left(1-\frac{\sqrt{69}}{9}\right), \quad \mu_{12} = \frac{16}{9}\left(1-\frac{325}{23\sqrt{69}}\right), \quad \mu_{22} = \frac{61}{81}\left(1-\frac{20091}{1403\sqrt{69}}\right).$$

$$\mu_{03} = \frac{22}{9\sqrt{69}}, \quad \mu_{13} = \frac{8800}{207\sqrt{69}}, \quad \mu_{23} = \frac{8296}{621}\sqrt{69}.$$

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