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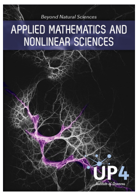

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Homoclinic and Heteroclinic Motions in Economic Models with Exogenous Shocks

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Abstract

In this study, we theoretically prove the presence of homoclinic and heteroclinic motions in the dynamics of economic models perturbed with exogenous shocks. An illustrative example based on the Kaldor model of the aggregate economy is presented.

Keywords: Exogenous shocks; Homoclinic and heteroclinic motions; Stable and unstable sets; Kaldor model.
AMS 2010 codes: 91G80, 35A24, 54H20.

1 Introduction

One of the most significant discoveries of H. Poincaré in the theory of dynamical systems is the presence of homoclinic orbits in the three body problem of celestial mechanics [1, 2]. In any neighborhood of a structurally stable Poincaré homoclinic orbit there exist nontrivial hyperbolic sets containing a countable number of saddle periodic orbits and continuum of non-periodic Poisson stable orbits [3–5]. Therefore, the existence of a structurally stable Poincaré homoclinic orbit can be considered as a criterion for the presence of chaos [3]. On the other hand, heteroclinic orbits are also important for the investigation of chaotic dynamics [6, 7].

The existence of homoclinic and heteroclinic motions in economic models has been extensively investigated in the literature [8–12]. The paper [8] deals with the occurrence of homoclinic and heteroclinic connections in a nonlinear overlapping generations (OLG) model with credit market imperfection and endogenous labor supply. By means of a singular perturbation method, the presence of transverse homoclinic points to the golden rule steady state in a two-dimensional Diamond-type OLG model was shown in [9]. In paper [10], the presence of homoclinic tangles associated with saddle points or saddle cycles of different period was demonstrated for a

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particular version of discrete-time Kaldor business cycle model. Homoclinic bifurcations in a class of models representing heterogeneous agents with adaptively rational rules was investigated within the scope of the paper [11]. Moreover, the existence of a homoclinic bifurcation was explored in [12] by using numerical simulations.

Chaos in the sense of Devaney [2] as well as the one obtained through period-doubling cascade [13] were investigated in our study [14] for economic models perturbed with exogenous shocks. It was shown in the paper [15] that exogenous shocks can cause economic models to exhibit chaotic business cycles. Other chaos generation techniques in systems of differential equations can be found in [16–26]. In the present study, we theoretically prove that exogenous shocks are capable of generating homoclinic and heteroclinic motions in the continuous-time dynamics of economic systems, and we numerically demonstrate the presence of such motions in the Kaldor model of the aggregate economy. The usage of exogenous shocks in the formation of homoclinic and heteroclinic motions is the main novelty of this paper.

Exogenous shocks in a macroeconomic model of a country can occur in two types [14]. The first one is the generation of shocks that are either completely outside of human control or are shaped in some worldwide marketplace. One can think of the economic fluctuations caused by weather phenomena, commodity prices that are determined in the world markets, and the futures prices of wheat, sugar, corn, soybean, coffee as well as oil product prices [27–29] as examples of the first type. The second type of exogenous shocks can be generated outside the economic system, but endogenous to some other system that is linked with the former through financial, trade and information flows. Exports to the foreign country may be viewed as an exogenous shock to the domestic economic system in the case that the real output in a foreign economy affects the level of demand by this economy for the exports of the home country, and exports to the foreign economy influence the economic activity at home [14].

The generation of homoclinic and heteroclinic motions in systems of ordinary differential equations by means of discontinuous perturbations was first considered in [30]. According to the results of [30], homoclinic solutions take place in the chaotic attractor of the relay system, which was introduced in [16]. On the other hand, by taking advantage of the moments of impulses, similar results were obtained for impulsive differential equations in [31].

The rest of the paper is organized as follows. In Section 2, we introduce the economic model that will be investigated in the present study. Section 3 is devoted to the theoretical results about the existence of homoclinic and heteroclinic solutions in the model. An example concerning the Kaldor model of the aggregate economy is presented in Section 4 in order to support the theoretical results. Finally, some concluding remarks are given in Section 5.

2 The model

Throughout the paper $\mathbb{R}$ and $\mathbb{Z}$ will stand for the sets of real numbers and integers, respectively. Moreover, the usual Euclidean norm for vectors and the norm induced by the Euclidean norm for square matrices [32] will be utilized.

In the present study, we take into account economic models perturbed with pulse functions. A function $p:\mathbb{R}\to\mathbb{R}^n$ is called a pulse function if for each integer i there is $p_i\in\mathbb{R}^n$ such that $p(t)=p_i$ either for $t\in(\theta_i,\theta_{i+1}]$ or for $t\in[\theta_i,\theta_{i+1})$, where $\{\theta_i\}_{i\in\mathbb{Z}}$ is a strictly increasing sequence of real numbers such that $|\theta_i|\to\infty$ as $|i|\to\infty$.

Consider the following economic model,

$$\dot{v}=H(v), \tag{1}$$

where the function $H:\mathbb{R}^n\to\mathbb{R}^n$ is continuously differentiable in its arguments and $v:\mathbb{R}\to\mathbb{R}^n$ is a function of time t. We assume that (1) possesses a steady state at $v=v_0$.

Let $[\tau]$ denote the largest integer that is not greater than $\tau\in\mathbb{R}$, and fix a positive number h. We perturb (1) with the pulse function $\widetilde{d}_{[t/h]}$, and set up the model

$$\dot{v} = H(v) + \widetilde{d}_{[t/h]}, \tag{2}$$

where $t \in \mathbb{R}$, $\widetilde{d}_i = (g(d_i), 0, 0, \ldots, 0) \in \mathbb{R}^n$ for each $i \in \mathbb{Z}$, the function $g : \Lambda \to \mathbb{R}$ is continuous, $\Lambda \subset \mathbb{R}$ is a bounded interval, and the sequence $\{d_i\}_{i\in\mathbb{Z}}$, $d_0 \in \Lambda$, is a solution of the discrete equation

$$d_{i+1} = F(d_i), \tag{3}$$

where $F : \Lambda \to \Lambda$ is a continuous function. It is worth noting that $\widetilde{d}_{[t/h]} = \widetilde{d}_i$ for $t \in [ih, (i+1)h)$.

Because of the economic reasons mentioned in Section 4, we consider the perturbation $\widetilde{d}_{[t/h]}$ with only one non-zero coordinate, and the more general case can be investigated in a similar way.

Exogenous shocks with variable values have many applications from the economic point of view. Economic time series such as commodity prices, productivity indices and international trade indicators are examples of exogenous shocks, and they are usually gauged by economists at regular discrete intervals, no matter how disaggregated (year, month, day, minute, second). Moreover, the government budget that is determined once a year, earnings of a farm that sells its produce in accordance with the seasons, and a firm's capital equipment that changes with periodical investment can be considered as other examples [14]. Our main purpose is to rigorously prove that homoclinic and heteroclinic motions exist in the dynamics of (2) by taking advantage of the exogenous shocks.

If we transform the state variables $x = v - v_0$ in (2), then near the equilibrium point the linearized model takes the form

$$\dot{x} = Ax + f(x) + \widetilde{d}_{[t/h]}, \tag{4}$$

where A is an $n \times n$ constant real valued matrix and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a function such that $f(0) = 0$. We suppose that all eigenvalues the matrix A have negative real parts.

The chaotic dynamics of the model under investigation was studied in the paper [14]. More precisely, it was theoretically proved in [14] that system (4) possesses chaos in the sense of Devaney [2] and through period-doubling cascade [13] provided that the same is true for the map (3). In the next section, we will show that homoclinic as well as heteroclinic motions take place in the continuous-time dynamics of (4) in the case that the map (3) possesses homoclinic and heteroclinic orbits.

3 Homoclinic and heteroclinic motions

According to the assumption that the eigenvalues of the matrix A in (4) have all negative real parts, there exist positive numbers N and ω such that $\left\|e^{At}\right\| \leq Ne^{-\omega t}$ for all $t \geq 0$.

The following conditions are required.

(C1) There exist positive numbers M_f and M_g such that $\sup_{x\in\mathbb{R}^n} \|f(x)\| \leq M_f$ and $\sup_{z\in\Lambda} |g(z)| \leq M_g$;

(C2) There exists a positive number $L_f < \omega/N$ such that $\|f(x_1) - f(x_2)\| \leq L_f \|x_1 - x_2\|$ for all $x_1, x_2 \in \mathbb{R}^n$;

(C3) There exists a positive number L_g such that $|g(z_1) - g(z_2)| \leq L_g |z_1 - z_2|$ for all $z_1, z_2 \in \Lambda$.

Let $\mathscr{D}$ be the set of all sequences $d = \{d_i\}_{i\in\mathbb{Z}}$ generated by the map (3). Under the conditions $(C1)$ and $(C2)$, for a given sequence $d = \{d_i\}_{i\in\mathbb{Z}} \in \mathscr{D}$, system (4) possesses a unique solution $\phi_d(t)$ which is bounded on $\mathbb{R}$ [33]. According to the results of [34], the bounded solution $\phi_d(t)$ satisfies the relation

$$\phi_d(t) = \int_{-\infty}^{t} e^{A(t-s)} (f(\phi_d(s)) + \widetilde{d}_{[s/h]}) ds.$$

Denote by $\mathscr{B}$ the set of all bounded solutions $\phi_d(t)$, $d \in \mathscr{D}$, of (4). It can be verified that $\sup_{t\in\mathbb{R}} \|\phi_d(t)\| \leq \dfrac{N(M_f+M_g)}{\omega}$ for each $\phi_d(t) \in \mathscr{B}$. For a given sequence $d \in \mathscr{D}$, if $x_d(t,x_0)$ is the solution of (4) satisfying $x_d(0,x_0) = x_0$, then we have

$$\|x_d(t,x_0) - \phi_d(t)\| \leq N\|x_0 - \phi_d(0)\| e^{(NL_f-\omega)t}$$

for all $t \geq 0$. Thus, $\|x_d(t,x_0) - \phi_d(t)\| \to 0$ as $t \to \infty$ in accordance with condition $(C2)$, i.e. the bounded solution $\phi_d(t)$ attracts all other solutions of (4) for a fixed $d \in \mathscr{D}$.

Now, let us continue with the definitions of the stable and unstable sets as well as the hyperbolicity for system (2) and the map (3). These definitions are adapted from the paper [30].

The stable set of a sequence $d = \{d_i\}_{i\in\mathbb{Z}} \in \mathscr{D}$ is defined as

$$W^s(d) = \left\{c = \{c_i\}_{i\in\mathbb{Z}} \in \mathscr{D} : \ \|c_i - d_i\| \to 0 \text{ as } i \to \infty\right\},$$

and the unstable set of d is

$$W^u(d) = \left\{c = \{c_i\}_{i\in\mathbb{Z}} \in \mathscr{D} : \ \|c_i - d_i\| \to 0 \text{ as } i \to -\infty\right\}.$$

The set $\mathscr{D}$ is called hyperbolic if for each $d \in \mathscr{D}$ the stable and unstable sets of d contain at least one element different from d. A sequence $c \in \mathscr{D}$ is homoclinic to another sequence $d \in \mathscr{D}$ if $c \in W^s(d) \cap W^u(d)$. Moreover, $c \in \mathscr{D}$ is heteroclinic to the sequences d^1, $d^2 \in \mathscr{D}$, $c \neq d^1$, $c \neq d^2$, if $c \in W^s(d^1) \cap W^u(d^2)$.

On the other hand, a bounded solution $\phi_c(t) \in \mathscr{B}$ belongs to the stable set $W^s(\phi_d(t))$ of $\phi_d(t) \in \mathscr{B}$ if $\|\phi_c(t) - \phi_d(t)\| \to 0$ as $t \to \infty$. Besides, $\phi_c(t)$ is an element of the unstable set $W^u(\phi_d(t))$ of $\phi_d(t)$ provided that $\|\phi_c(t) - \phi_d(t)\| \to 0$ as $t \to -\infty$.

We say that the set $\mathscr{B}$ is hyperbolic if for each $\phi_d(t) \in \mathscr{B}$ the sets $W^s(\phi_d(t))$ and $W^u(\phi_d(t))$ contain at least one element different from $\phi_d(t)$. A solution $\phi_c(t) \in \mathscr{B}$ is homoclinic to another solution $\phi_d(t) \in \mathscr{B}$ if $\phi_c(t) \in W^s(\phi_d(t)) \cap W^u(\phi_d(t))$, and $\phi_c(t) \in \mathscr{B}$ is heteroclinic to the bounded solutions $\phi_{d^1}(t)$, $\phi_{d^2}(t) \in \mathscr{B}$, $\phi_c(t) \neq \phi_{d^1}(t)$, $\phi_c(t) \neq \phi_{d^2}(t)$, if $\phi_c(t) \in W^s(\phi_{d^1}(t)) \cap W^u(\phi_{d^2}(t))$.

In the next lemma, we deal with the connection between the stable sets of the solutions of (3) and (4).

Lemma 1. *Suppose that the conditions $(C1)-(C3)$ hold, and let $c = \{c_i\}_{i\in\mathbb{Z}}$ and $d = \{d_i\}_{i\in\mathbb{Z}}$ be elements of $\mathscr{D}$. If $c \in W^s(d)$, then $\phi_c(t) \in W^s(\phi_d(t))$.*

Proof. Fix an arbitrary positive number ε, and let γ be a number such that $\gamma \geq 1 + \dfrac{NL_g}{\omega - NL_f}$. Since $c \in W^s(d)$, there exists an integer j_0 such that $\left|c_{j_0} - d_{j_0}\right| < \dfrac{\varepsilon}{\gamma}$ for all $i \geq j_0$. In this case, we have $\left\|\widetilde{c}_{[t/h]} - \widetilde{d}_{[t/h]}\right\| < \dfrac{L_g\varepsilon}{\gamma}$ for $t \geq j_0 h$.

By using the relation

$$\phi_c(t) - \phi_d(t) = \int_{-\infty}^{t} e^{A(t-s)}\left[f(\phi_c(s)) + \widetilde{c}_{[s/h]} - f(\phi_d(s)) - \widetilde{d}_{[s/h]}\right] ds,$$

one can obtain for $t \geq j_0 h$ that

$$\begin{aligned}\|\phi_c(t) - \phi_d(t)\| \leq{}& \frac{2N(M_f+M_g)}{\omega} e^{-\omega(t-j_0h)} + \frac{NL_g\varepsilon}{\omega\gamma}\left(1 - e^{-\omega(t-j_0h)}\right) \\ &+ \int_{j_0h}^{t} NL_f e^{-\omega(t-s)}\|\phi_c(s) - \phi_d(s)\| ds.\end{aligned}$$

If we denote $u(t) = e^{\omega t}\|\phi_c(t) - \phi_d(t)\|$ and $\alpha = \left(\frac{2N(M_f + M_g)}{\omega} - \frac{NL_g\varepsilon}{\omega\gamma}\right)e^{\omega j_0 h}$, then we attain

$$u(t) \leq \alpha + \frac{NL_g\varepsilon}{\omega\gamma}e^{\omega t} + \int_{j_0h}^{t} NL_f u(s)ds.$$

It can be verified by applying the Gronwall's Lemma [33] that

$$u(t) \leq \alpha + \frac{NL_g\varepsilon}{\omega\gamma}e^{\omega t} + \int_{j_0h}^{t} NL_f\left(\alpha + \frac{NL_g\varepsilon}{\omega\gamma}e^{\omega s}\right)e^{NL_f(t-s)}ds.$$

Therefore,

$$u(t) \leq \frac{NL_g\varepsilon}{\omega\gamma}e^{\omega t} + \alpha e^{NL_f(t-j_0h)} + \frac{N^2L_fL_g\varepsilon}{\omega(\omega - NL_f)\gamma}e^{\omega t}\left(1 - e^{-(\omega - NL_f)(t-j_0h)}\right).$$

The last inequality yields

$$\|\phi_c(t) - \phi_d(t)\| < \frac{NL_g\varepsilon}{(\omega - NL_f)\gamma} + \frac{2N(M_f + M_g)}{\omega}e^{(NL_f - \omega)(t-j_0h)},\ t \geq j_0h.$$

Now, let R be a sufficiently large positive number such that

$$\frac{2N(M_f + M_g)}{\omega}e^{(NL_f - \omega)R} \leq \frac{\varepsilon}{\gamma}.$$

For $t \geq R + j_0h$, we have that

$$\|\phi_c(t) - \phi_d(t)\| < \frac{\varepsilon}{\gamma}\left(1 + \frac{NL_g}{\omega - NL_f}\right) \leq \varepsilon.$$

Consequently, $\phi_c(t)$ belongs to the stable set $W^s(\phi_d(t))$ of $\phi_d(t) \in \mathscr{B}$. □

The following assertion is concerned with the unstable sets of the solutions of (3) and (4).

Lemma 2. *Suppose that the conditions* $(C1)-(C3)$ *hold, and let* $c = \{c_i\}_{i\in\mathbb{Z}}$ *and* $d = \{d_i\}_{i\in\mathbb{Z}}$ *be elements of* $\mathscr{D}$. *If* $c \in W^u(d)$, *then* $\phi_c(t) \in W^u(\phi_d(t))$.

Proof. Fix an arbitrary positive number ε, and let γ be a number such that $\gamma > \frac{NL_g}{\omega - NL_f}$. One can find an integer j_0 such that $|c_i - d_i| < \frac{\varepsilon}{\gamma}$ for all $i \leq j_0$ since the sequence c belongs to the unstable set $W^u(d)$ of $d \in \mathscr{D}$. Hence, $\left\|\widetilde{c}_{[t/h]} - \widetilde{d}_{[t/h]}\right\| < \frac{L_g\varepsilon}{\gamma}$ for $t < (j_0 + 1)h$.

Making use of the equation

$$\phi_c(t) - \phi_d(t) = \int_{-\infty}^{t} e^{A(t-s)}\left[f(\phi_c(s)) + \widetilde{c}_{[s/h]} - f(\phi_d(s)) - \widetilde{d}_{[s/h]}\right]ds,$$

it can be obtained for $t < (j_0 + 1)h$ that

$$\|\phi_c(t) - \phi_d(t)\| < \frac{NL_g\varepsilon}{\omega\gamma} + \int_{-\infty}^{t} NL_f e^{-\omega(t-s)}\|\phi_c(s) - \phi_d(s)\|ds.$$

Therefore,

$$\sup_{t<(j_0+1)h}\|\phi_c(t) - \phi_d(t)\| \leq \frac{NL_g\varepsilon}{(\omega - NL_f)\gamma} < \varepsilon.$$

Consequently, $\|\phi_c(t) - \phi_d(t)\| \to 0$ as $t \to -\infty$ so that $\phi_c(t) \in W^u(\phi_d(t))$. □

The following theorem, which is the main result of the present paper, can be proved by using Lemmas 1 and 2.

Theorem 3. *Under the conditions* $(C1)-(C3)$, *the following assertions are valid.*

(i) If $c \in \mathscr{D}$ *is homoclinic to* $d \in \mathscr{D}$, *then* $\phi_c(t) \in \mathscr{B}$ *is homoclinic to* $\phi_d(t) \in \mathscr{B}$;

(ii) If $c \in \mathscr{D}$ *is heteroclinic to* $d^1, d^2 \in \mathscr{D}$, *then* $\phi_c(t) \in \mathscr{B}$ *is heteroclinic to* $\phi_{d^1}(t), \phi_{d^2}(t) \in \mathscr{B}$;

(iii) If $\mathscr{D}$ *is hyperbolic, then the same is true for* $\mathscr{B}$.

An example concerning the Kaldor model of the aggregate economy will be taken into account in the next section.

4 An example

Consider the following Kaldor model [35,36],

$$\begin{aligned} \dot{Y} &= \alpha[I(Y,K) - S(Y,K)] \\ \dot{K} &= I(Y,K) - \delta K, \end{aligned} \tag{5}$$

where $\delta \in (0,1)$ is the constant depreciation rate, $\alpha > 0$ is the adjustment coefficient, Y is income, K is capital stock, I is gross investment and S is savings.

Let us use $I(Y,K) = Y - aY^3 + bK$ and $S(Y,K) = sY$ in (5) so that the system takes the form

$$\begin{aligned} \dot{Y} &= \alpha[(1-s)Y - aY^3 + bK] \\ \dot{K} &= Y - aY^3 + (b-\delta)K, \end{aligned} \tag{6}$$

where the constant parameters satisfy $a > 0$, $b < 0$ and $0 < s < 1$. In the case that $s(b-\delta)+\delta > 0$, system (6) admits the following steady state with positive coordinates:

$$Y^* = \sqrt{\frac{s(b-\delta)+\delta}{a\delta}}, \quad K^* = \frac{s}{\delta}\sqrt{\frac{s(b-\delta)+\delta}{a\delta}}.$$

The income Y of a given country is subject to many possible exogenous disturbances, such as productivity shocks and global economic fluctuations, while the capital stock K can be viewed as a mechanical relation between investment and capital stock, where there is little room for exogenous influences. Therefore, we modify (6) by using perturbation only in the equation for income Y, and constitute the system

$$\begin{aligned} \dot{Y} &= \alpha[(1-s)Y - aY^3 + bK] + g(d_{[t]}) \\ \dot{K} &= Y - aY^3 + (b-\delta)K, \end{aligned} \tag{7}$$

where the function g is defined as $g(z) = 0.02(z - 0.4\sin z)$. The sequence $\{d_i\}_{i\in\mathbb{Z}}$, $d_0 \in [0,1]$, is a solution of the logistic map

$$d_{i+1} = F(d_i), \tag{8}$$

where $F(\sigma) = 3.8\sigma(1-\sigma)$. Notice that $d_{[t]} = d_i$ for $i \le t < i+1$, $i \in \mathbb{Z}$. The unit interval $[0,1]$ is invariant under the iterations of (8) [37]. Moreover, the inverses of the function F on the intervals $[0,1/2]$ and $[1/2,1]$ are $G_1(\sigma) = \dfrac{1}{2}\left(1 - \sqrt{1 - \dfrac{4\sigma}{3.8}}\right)$ and $G_2(\sigma) = \dfrac{1}{2}\left(1 + \sqrt{1 - \dfrac{4\sigma}{3.8}}\right)$, respectively. For the applications of the logistic map the reader is referred to [38].

In what follows, we will make use of the values $\alpha = 1$, $a = 0.01$, $b = -1/12$, $s = 1/3$ and $\delta = 1/6$. Using the transformation $y = Y - Y^*$, $k = K - K^*$ in (7), where $Y^* = 5\sqrt{2}$, $K^* = 10\sqrt{2}$, we obtain the system

$$\begin{aligned} \dot{y} &= -\frac{5}{6}y - \frac{1}{12}k - 0.01y^3 - \frac{3}{10\sqrt{2}}y^2 + g(d_{[t]}) \\ \dot{k} &= -\frac{1}{2}y - \frac{1}{4}k - 0.01y^3 - \frac{3}{10\sqrt{2}}y^2. \end{aligned} \tag{9}$$

System (9) is in the form of (4), where

$$A = \begin{pmatrix} -\frac{5}{6} & -\frac{1}{12} \\ -\frac{1}{2} & -\frac{1}{4} \end{pmatrix}, \quad f(y,k) = \begin{pmatrix} -0.01y^3 - \frac{3}{10\sqrt{2}}y^2 \\ -0.01y^3 - \frac{3}{10\sqrt{2}}y^2 \end{pmatrix}.$$

The eigenvalues of the matrix A are $\frac{-13+\sqrt{73}}{24}$ and $\frac{-13-\sqrt{73}}{24}$. One can confirm that $e^{At} = Pe^{Dt}P^{-1}$, where

$$P = \begin{pmatrix} \frac{7-\sqrt{73}}{12} & 1 \\ 1 & \frac{-7+\sqrt{73}}{2} \end{pmatrix}, \quad D = \begin{pmatrix} \frac{-13+\sqrt{73}}{24} & 0 \\ 0 & \frac{-13-\sqrt{73}}{24} \end{pmatrix}.$$

Thus, $\left\|e^{At}\right\| \leq Ne^{-\omega t}$ for $t \geq 0$, where $N = \|P\| \left\|P^{-1}\right\| \approx 1.83005$ and $\omega = \frac{13-\sqrt{73}}{24}$.

It can be numerically verified that the solutions of (9) which are bounded on the entire real axis lie inside the compact region

$$\mathscr{U} = \left\{(y,k) \in \mathbb{R}^2 : 0.008 \leq y \leq 0.016,\ -0.022 \leq k \leq -0.028\right\}.$$

Therefore, it is reasonable to consider the conditions $(C1)$ and $(C2)$ for the function $f(y,k)$ on the region $\mathscr{U}$ so that $(C2)$ is valid with $L_f = 0.0097$. On the other hand, $(C3)$ holds with $L_g = 0.0157$.

Since the inequality $|g(z_1) - g(z_2)| \geq 0.012\,|z_1 - z_2|$, $z_1, z_2 \in [0,1]$, is also valid in addition to $(C1)-(C3)$, system (9) (and hence (7)) admits the chaos through period-doubling cascade according to the results of [14]. In order to demonstrate the presence of chaos, let us consider the solution of (9) with $y(0) = 0.014$, $k(0) = -0.025$ and $d_0 = 0.18$. The trajectory of the solution is depicted in Figure 1, which reveals that (9) is chaotic.

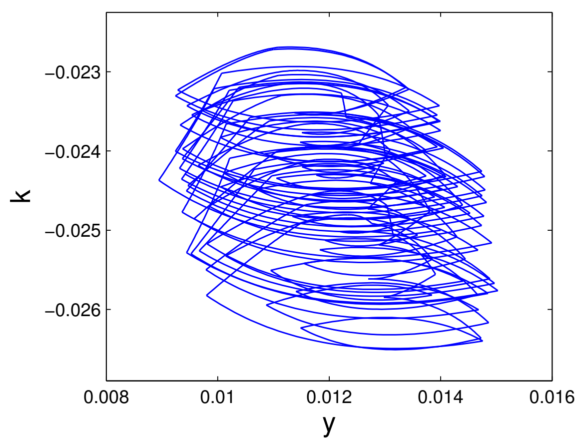

Fig. 1 The trajectory of system (7) corresponding to the initial data $y(0) = 0.014$ and $k(0) = -0.025$. The solution of (8) with $d_0 = 0.18$ is used in the simulation.

Now, we will show the presence of homoclinic and heteroclinic motions in the dynamics of (9). It was mentioned in the paper [39] that the orbit $c = \left\{\ldots, G_2^3(c_0), G_2^2(c_0), G_2(c_0), c_0, F(c_0), F^2(c_0), F^3(c_0), \ldots\right\}$, where $c_0 = 1/3.8$, is homoclinic to the fixed point $d^* = 2.8/3.8$ of (8). Let $\phi_c(t)$ and $\phi_{d^*}(t)$ be the bounded on $\mathbb{R}$ solutions of (9) corresponding to c and d^*, respectively. The solution $\phi_c(t)$ is homoclinic to $\phi_{d^*}(t)$ in accordance with Theorem 3. Figure 2 shows the graphs of $\phi_c(t)$ and $\phi_{d^*}(t)$ in blue and red colors, respectively.

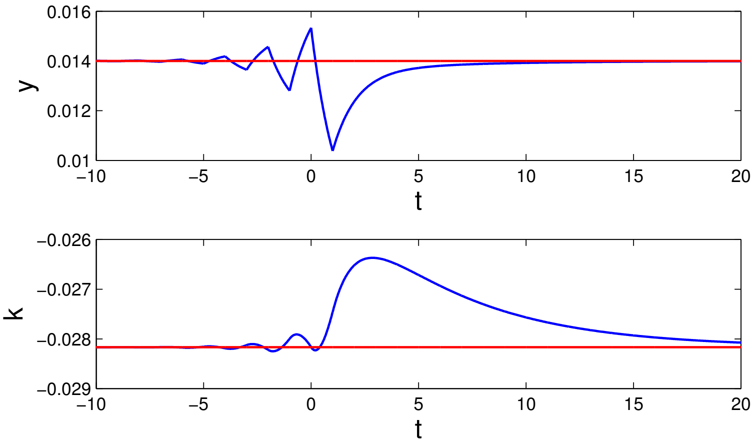

Fig. 2 The homoclinic solution of (9). The bounded solutions $\phi_c(t)$ and $\phi_{d^*}(t)$ are depicted in blue and red colors, respectively. It seen in the figure that $\phi_c(t)$ is homoclinic to $\phi_{d^*}(t)$.

Both the y and k coordinates of the solutions are represented in the figure. The simulation results confirm that $\|\phi_c(t) - \phi_{d^*}(t)\| \to 0$ as $t \to \pm\infty$, that is, $\phi_c(t)$ is homoclinic to $\phi_{d^*}(t)$.

Next, we take into account the orbit $\overline{c} = \left\{\ldots, G_1^3(\overline{c}_0), G_1^2(\overline{c}_0), G_1(\overline{c}_0), \overline{c}_0, F(\overline{c}_0), F^2(\overline{c}_0), F^3(\overline{c}_0), \ldots\right\}$, where $\overline{c}_0 = 1/3.8$. According to [39], the orbit $\overline{c}$ is heteroclinic to the fixed points $d^* = 2.8/3.8$ and $d^{**} = 0$ of (8). One can conclude by using Theorem 3 that $\phi_{\overline{c}}(t)$ is heteroclinic to $\phi_{d^*}(t)$ and $\phi_{d^{**}}(t)$, where $\phi_{\overline{c}}(t)$, $\phi_{d^*}(t)$ and $\phi_{d^{**}}(t)$ are respectively the bounded on $\mathbb{R}$ solutions of (9) corresponding to $\overline{c}$, d^* and d^{**}. Figure 3 represents the y and k coordinates of $\phi_{\overline{c}}(t)$, $\phi_{d^*}(t)$ and $\phi_{d^{**}}(t)$ in blue, red and green colors, respectively. The figure supports Theorem 3 such that $\phi_{\overline{c}}(t)$ is heteroclinic to $\phi_{d^*}(t)$, $\phi_{d^{**}}(t)$.

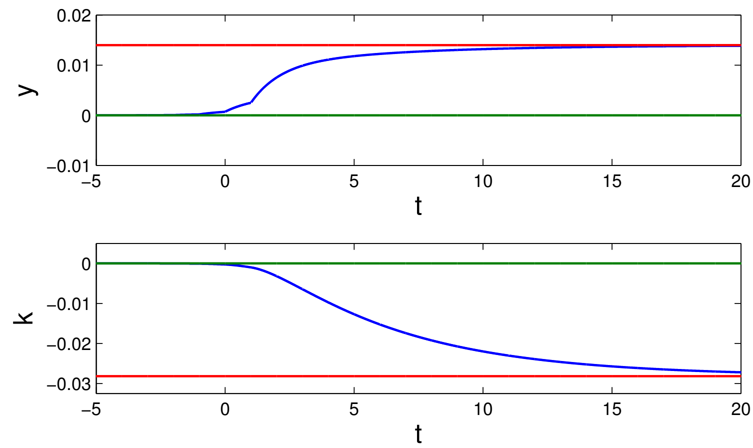

Fig. 3 The heteroclinic solution of (9). The bounded solutions $\phi_{\overline{c}}(t)$, $\phi_{d^*}(t)$ and $\phi_{d^{**}}(t)$ are depicted in blue, red and green colors, respectively. The simulation demonstrates that $\phi_{\overline{c}}(t)$ is heteroclinic to $\phi_{d^*}(t)$, $\phi_{d^{**}}(t)$.

5 Conclusions

Homoclinic and heteroclinic orbits are crucial in the theory of dynamical systems since their presence is related to the existence of chaos. We provide a theoretical approach for the generation of homoclinic and heteroclinic motions in the continuous-time dynamics of economic models influenced by exogenous shocks. The main novelty of the present paper is the formation of such motions *exogenously*. The example concerning the Kaldor model of the aggregate economy presented in Section 4 supports the theoretical results such that exogenous shocks in economic models can lead to the occurrence of homoclinic and heteroclinic motions. The presented technique can be useful for the investigation of homoclinic and heteroclinic bifurcations in economic systems influenced by exogenous shocks.

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