

## DECENTRALIZED STABILIZATION OF FRACTIONAL POSITIVE DESCRIPTOR CONTINUOUS-TIME LINEAR SYSTEMS

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A method for decentralized stabilization of fractional positive descriptor linear systems is proposed. Necessary and sufficient conditions for decentralized stabilization of fractional positive descriptor linear systems are established. The efficiency of the proposed method is demonstrated on a numerical example.

**Keywords:** positive system, linear system, continuous-time system, descriptor system, decentralized stabilization.

### 1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of the state of the art in positive systems theory was given by Farina and Rinaldi (2000) or Kaczorek (2002; 2014a; 2014b; 2010). Models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

A dynamical system is called fractional if it is described by fractional differential equations. Mathematical fundamentals of fractional calculus were given and by Oldham and Spanier (1974), Ostalczyk (2008) and Podlubny (1999). Positive fractional linear systems were investigated by Caputo and Fabrizio (2015), Kaczorek (2013; 2010; 2012), or Losada and Nieto (2015). Stability of positive descriptor systems was addressed by Kaczorek (2011a) and Virnik (2008).

Descriptor (singular) linear systems were considered in many papers and books (Bru *et al.*, 2000; 2003; State, 1976; Dai, 1989; Dodig and Stosic, 2009; Duan, 2010; Fahmy and o'Reill, 1989; Sajewski, 2016a; 2016b). Positive standard and descriptor systems and their stability were analyzed by Dodig and Stosic (2009), Duan (2010), or Sajewski (2016b). Descriptor standard and positive discrete-time and continuous-time nonlinear systems were analyzed by Kaczorek (2014a; 2012; 1997) and Van Dooren (1979). The Drazin inverse was applied to the analysis of descriptor systems by

Kaczorek (2013), who also addressed singular standard and positive linear systems (Kaczorek, 2002; 1997), along with minimum energy control of descriptor positive systems (Kaczorek, 2014b) and positive linear systems with different fractional orders (Kaczorek, 2010; 2011b). Positive fractional continuous-time linear systems with singular pencil were also addressed by Kaczorek (2012).

This work is intended as an attempt to characterize decentralized stabilization of fractional positive descriptor continuous-time linear systems. The paper is organized as follows. In Section 2 some definitions and theorems concerning fractional positive linear continuous-time systems are recalled and a method of stabilization of linear systems by state-feedback is proposed. The main result of the paper, a method of decentralized stabilization of fractional descriptor linear systems, is presented in Section 3. In Section 4, concluding remarks are given.

The following notation will be used:  $\mathbb{R}$ , the set of real numbers;  $\mathbb{R}^{n \times m}$ , the set of real  $n \times m$  matrices;  $\mathbb{R}_+^{n \times m}$ , the set of  $n \times m$  matrices with nonnegative entries and  $\mathbb{R}_+^n = \mathbb{R}_+^{n \times 1}$ ;  $\mathbb{Z}_+$ , the set of nonnegative integers;  $\mathbb{M}_n$ , the set of  $n \times n$  Metzler matrices (with nonnegative off-diagonal entries);  $I_n$ , the  $n \times n$  identity matrix;  $A^T$ , the transposition of matrix  $A$ .

### 2. Preliminaries

In this paper, the following Caputo definition of the fractional derivative of  $\alpha$  order will be used (Kaczorek, 2011a; Oldham and Spanier, 1974; Ostalczyk, 2008;

Podlubny, 1999):

$${}_0D_t^\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{f}(\tau)}{(t-\tau)^\alpha} d\tau, \quad 0 < \alpha < 1, \quad (1)$$

where

$$\dot{f}(\tau) = \frac{df(\tau)}{d\tau}$$

and

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt,$$

while  $\Gamma(x) > 0$  is the Euler gamma function.

Consider the fractional continuous-time linear system

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bu(t), \quad 0 < \alpha < 1, \quad (2a)$$

$$y(t) = Cx(t), \quad (2b)$$

where  $x(t) \in \mathbb{R}^n$ ,  $u(t) \in \mathbb{R}^m$ ,  $y(t) \in \mathbb{R}^p$  are the state, input and output vectors, respectively, and  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $C \in \mathbb{R}^{p \times n}$ .

**Definition 1.** (Farina and Rinaldi, 2000; Kaczorek, 2002)

The fractional system (2) is called (*internally*) *positive* if  $x(t) \in \mathbb{R}_+^n$ ,  $y(t) \in \mathbb{R}_+^p$ ,  $t \geq 0$  for all  $x(0) \in \mathbb{R}_+^n$  and every  $u(t) \in \mathbb{R}_+^m$ ,  $t \geq 0$ .

**Theorem 1.** (Farina and Rinaldi, 2000; Kaczorek, 2002)  
The fractional system (2) is positive if and only if

$$A \in \mathbb{M}_n, \quad B \in \mathbb{R}_+^{n \times m}, \quad C \in \mathbb{R}_+^{p \times n}. \quad (3)$$

**Definition 2.** (Farina and Rinaldi, 2000; Kaczorek, 2011a) The positive fractional system (2) for  $u(t) = 0$  is called asymptotically stable if

$$\lim_{t \rightarrow \infty} x(t) = 0 \quad \text{for all } x(0) \in \mathbb{R}_+^n. \quad (4)$$

**Theorem 2.** (Farina and Rinaldi, 2000; Kaczorek, 2011a)  
The positive fractional system (2) for  $u(t) = 0$  is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1. All coefficients of the characteristic polynomial

$$\det[I_n s - A] = s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 \quad (5)$$

are positive, i.e.,  $a_i > 0$ ,  $i = 0, 1, \dots, n-1$ .

2. All principal minors  $\bar{M}_i > 0$ ,  $i = 1, \dots, n$ , of the matrix  $-A$  are positive, i.e.,

$$\begin{aligned} \bar{M}_1 &= |-a_{11}| > 0, \\ \bar{M}_2 &= \begin{vmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{vmatrix} > 0, \\ \bar{M}_n &= \det[-A] > 0. \end{aligned} \quad (6)$$

3. There exists a strictly positive vector

$$\lambda = [\lambda_1, \dots, \lambda_n]^T, \quad \lambda_k > 0, \quad k = 1, \dots, n,$$

such that

$$A\lambda < 0. \quad (7)$$

If  $\det A \neq 0$  then we may choose  $\lambda = A^{-1}c$ , where  $c \in \mathbb{R}^n$  is strictly positive.

We are looking for a gain matrix

$$K = ND^{-1}, \quad D = \text{diag}[d_1, \dots, d_n], \quad d_k > 0, \quad k = 1, \dots, n, \quad N \in \mathbb{R}^{m \times n} \quad (8)$$

such that

$$A + BK \in \mathbb{M}_n \quad (9)$$

is asymptotically stable.

Substitution of (8) into (9) yields

$$A + BK = (AD + BN)D^{-1}. \quad (10)$$

Using the known (Boyd *et al.*, 1994; Giorgio and Zuccotti, 2015) procedures we choose the matrices  $D$  and  $N$  so that

$$AD + BN \in \mathbb{M}_n, \quad (AD + BN)D^{-1} < 0. \quad (11)$$

If (11) holds, then the matrix (10) is an asymptotically stable Metzler matrix.

To find the matrices  $D$  and  $N$ , one of the well-known linear programming or LMI procedures can be used (Boyd *et al.*, 1994; Giorgio and Zuccotti, 2015).

**Definition 3.** The positive system (1) (or, equivalently, the pair  $(A, B)$ ) is called *stabilizable by the state feedback* if there exists a gain matrix (8) such that the closed-loop system matrix

$$A_c = A + BK \in \mathbb{M}_n \quad (12)$$

is asymptotically stable.

**Remark 1.** In a general case, the controllability of the pair  $(A, B)$  is not sufficient for the stabilization of a close-loop system with a Metzler matrix. The pair  $(A, B)$  should be stabilizable.

### 3. Decentralized stabilization of fractional descriptor linear systems

Consider the fractional descriptor linear continuous-time system

$$E \frac{d^\alpha x}{dt^\alpha} = Ax + Bu, \quad (13)$$

where  $x = x(t) \in \mathbb{R}^n$ ,  $u = u(t) \in \mathbb{R}^m$  are respectively the state and input vectors, and  $A, E \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ .

**Assumption 1.** The matrix  $E$  possesses  $n_1 < n$  linearly independent columns (the remaining columns are zero).

**Assumption 2.** The pencil of (13) is regular, i.e.,

$$\det[Es - A] \neq 0 \quad \text{for some } s \in \mathbb{C}, \quad (14)$$

where  $\mathbb{C}$  is the field of complex numbers.

Defining the new state vector

$$\begin{aligned} \bar{x} = \bar{x}(t) = P^{-1}x(t) &= \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}, \\ \bar{x}_1 &\in \mathbb{R}^{n_1}, \quad \bar{x}_2 \in \mathbb{R}^{n_2}, \quad n_2 = n - n_1 \end{aligned} \quad (15)$$

and premultiplying Eqn. (13) by a matrix  $Q \in \mathbb{R}^{n \times n}$ , we obtain

$$QEPP^{-1}\dot{x} = QAPP^{-1}x + QBu \quad (16)$$

and

$$\frac{d^\alpha x_1}{dt^\alpha} = \bar{A}_{11}x_1 + \bar{A}_{12}x_2 + \bar{B}_1u, \quad (17)$$

$$0 = \bar{A}_{21}x_1 + \bar{A}_{22}x_2 + \bar{B}_2u, \quad (18)$$

where

$$QEP = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}, \quad (19a)$$

$$\bar{A} = QAP = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \quad (19b)$$

$$\bar{A}_{11} \in \mathbb{R}^{n_1 \times n_1}, \quad \bar{A}_{22} \in \mathbb{R}^{n_2 \times n_2}, \quad (19c)$$

$$\bar{B} = QB = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix}, \quad (19d)$$

$$\bar{B}_1 \in \mathbb{R}^{n_1 \times m}, \quad \bar{B}_2 \in \mathbb{R}^{n_2 \times m}. \quad (19e)$$

The matrices  $Q$  and  $P$  can be obtained with the use of the following elementary row and column operations (Kaczorek, 2002; 2011a):

1. Multiplication of the  $i$ -th row (resp. column) by a real number  $c$ . This operation will be denoted by  $L[i \times c]$  (resp.  $R[i \times c]$ ).
2. Addition to the  $i$ -th row (resp. column) of the  $j$ -th row (resp. column) multiplied by a real number  $c$ . This operation will be denoted by  $L[i + j \times c]$  (resp.  $R[i + j \times c]$ ).
3. Interchange of the  $i$ -th and  $j$ -th rows (resp. columns). This operation will be denoted by  $L[i, j]$  (resp.  $R[i, j]$ ).

From Assumption 1 it follows that the matrix  $P$  is a permutation matrix and  $P^{-1} = P^T \in \mathbb{R}_+^{n \times n}$ . Therefore, if  $x(t) \in \mathbb{R}_+^n$ ,  $t \geq 0$ , then  $P^{-1}x(t) \in \mathbb{R}_+^n$ ,  $t \geq 0$ . The matrix  $Q \in \mathbb{R}^{n \times n}$  can be obtained by performing an elementary row operation on the identity matrix  $I_n$ .

Consider the fractional descriptor unstable positive linear system

$$\begin{aligned} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \frac{d^\alpha}{dt^\alpha} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \end{aligned} \quad (20a)$$

where

$$x_1 \in \mathbb{R}^{n_1}, \quad x_2 \in \mathbb{R}^{n_2}, \quad u \in \mathbb{R}^m \text{ and}$$

$$A_{11} \in M_{n_1}, \quad A_{22} \in M_{n_2}, \quad A_{12} \in \mathbb{R}_+^{n_1 \times n_2}, \quad (20b)$$

$$A_{21} \in \mathbb{R}_+^{n_2 \times n_1}, \quad B_1 \in \mathbb{R}_+^{n_1 \times m}, \quad B_2 \in \mathbb{R}_+^{n_2 \times m}. \quad (20c)$$

We are looking for a decentralized controller,

$$\begin{aligned} u = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\ K_1 \in \mathbb{R}^{n_1 \times n_1}, \quad K_2 \in \mathbb{R}^{n_2 \times n_2}, \end{aligned} \quad (21)$$

such that the closed-loop system

$$\begin{aligned} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \frac{d^\alpha x}{dt^\alpha} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ = \begin{bmatrix} A_{11} + B_1K_1 & A_{12} \\ A_{21} & A_{22} + B_2K_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \quad (22)$$

is positive and asymptotically stable.

The problem can be solved with the use of the following procedure.

**Procedure 1.**

*Step 1.* Given  $A_{22} \in M_{n_2}$  and  $B_2 \in \mathbb{R}_+^{n_2 \times m}$ , compute a gain matrix  $K_2$  such that

$$\hat{A}_{22} = A_{22} + B_2K_2 \in \mathbb{M}_{n_2} \quad (23)$$

and is asymptotically stable.

Then, from (22), we have

$$\begin{aligned} x_2 = -\hat{A}_{22}^{-1}A_{21}x_1 \in \mathbb{R}_+^{n_2} \\ \iff x_1 \in \mathbb{R}_+^{n_1}, \quad t \geq 0, \end{aligned} \quad (24)$$

where  $-\hat{A}_{22}^{-1}A_{21} \in \mathbb{R}_+^{n_2 \times n_1}$  since, by Theorem 2,  $-\hat{A}_{22}^{-1} \in \mathbb{R}_+^{n_2 \times n_2}$  and  $A_{21} \in \mathbb{R}_+^{n_2 \times n_1}$ .

Substituting (24) into

$$\hat{x}_1 = (A_{11} + B_1K_1)x_1 + A_{12}x_2, \quad (25)$$

we obtain

$$\dot{x}_1 = (\hat{A}_{11} + B_1K_1)x_1, \quad (26)$$

where

$$\hat{A}_{11} = A_{11} - A_{12}\hat{A}_{22}^{-1}A_{21} \in \mathbb{R}_+^{n_1 \times n_1}. \quad (27)$$

*Step 2.* Knowing  $\hat{A}_{11} \in \mathbb{R}_+^{n_1 \times n_1}$  and  $B \in \mathbb{R}_+^{n_1 \times m}$ , compute a gain matrix  $K_1$  such that

$$\hat{A}_{11} + B_1 K_1 \in \mathbb{M}_{n_1} \quad (28)$$

is asymptotically stable.

If the condition (28) is satisfied, then from (26) we have

$$x_1 \in \mathbb{R}_+^{n_1}, \quad \lim_{t \rightarrow \infty} x_1(t) = 0, \quad (29)$$

which implies

$$x_2 \in \mathbb{R}_+^{n_2}, \quad \lim_{t \rightarrow \infty} x_2(t) = 0. \quad (30)$$

It is well known that the eigenvalues of the matrix  $A_c = A + BK$  can be arbitrarily assigned to the state-feedback matrix  $K$  if and only if the pair  $(A, B)$  is stabilizable. Therefore, the following theorem has been proved.

**Theorem 3.** *The positive descriptor system (20) can be stabilized by the decentralized controller (21) if and only if the pairs  $(A_{22}, B_2)$ ,  $(\hat{A}_{11}, B_1)$  are stabilizable.*

**Remark 2.** Note that the closed-loop system is positive and asymptotically stable if the gain matrices  $K_1$  and  $K_2$  are chosen so that the matrices  $A_{22} + B_2 K_2$  and  $\hat{A}_{11} + B_1 K_1$  are asymptotically stable Metzler matrices, but the matrix

$$\begin{bmatrix} A_{11} + B_1 K_1 & A_{12} \\ A_{21} & A_{22} + B_2 K_2 \end{bmatrix} \quad (31)$$

is not necessarily asymptotically stable.

**Example 1.** Consider the positive descriptor system (20) with the matrices

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & -3 & 0 & 1 \\ 1 & 1 & -2 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix}, \quad (32)$$

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (n_1 = n_2 = 2).$$

The system is unstable since the matrices  $A_{11}$  and  $A_{22}$  have positive diagonal entries. ♦

Find a decentralized controller (21) which is stabilized for the system (20) with (32).

Using Procedure 1, we obtain what follows.

*Step 1.* Note that the pair

$$(A_{22}, B_2) = \left( \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

is controllable, and using (8) we obtain that

$$\begin{aligned} K_2 &= N_2 D_2^{-1} \\ &= \begin{bmatrix} 2 & -3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -3 \end{bmatrix} \end{aligned} \quad (33)$$

and the matrix

$$\begin{aligned} \hat{A}_{22} &= A_{22} + B_2 K_2 \\ &= \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -3 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \in \mathbb{M}_2 \end{aligned} \quad (34)$$

is asymptotically stable.

Taking into account that

$$-\hat{A}_{22}^{-1} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad (35)$$

we obtain

$$\begin{aligned} \hat{A}_{11} &= A_{11} - A_{12} \hat{A}_{22}^{-1} A_{21} \\ &= \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 5 & 10 \\ 4 & -4 \end{bmatrix}. \end{aligned} \quad (36)$$

Note that the matrix (36) is unstable and the pair  $(\hat{A}_{11}, B_1)$  is controllable.

*Step 2.* Using (8), we obtain that

$$\begin{aligned} K_1 &= N_1 D_1^{-1} \\ &= \begin{bmatrix} -4 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -4 & -3 \end{bmatrix} \end{aligned} \quad (37)$$

and the matrix

$$\begin{aligned} \hat{A}_{11} + B_1 K_1 &= \frac{1}{3} \begin{bmatrix} 5 & 10 \\ 4 & -4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} -4 & -3 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} -7 & 1 \\ 4 & -4 \end{bmatrix} \in \mathbb{M}_2 \end{aligned} \quad (38)$$

is asymptotically stable.

Note that the closed-loop system (22) with the matrix

$$\begin{aligned} &\begin{bmatrix} A_{11} + B_1 K_1 & A_{12} \\ A_{21} & A_{22} + B_2 K_2 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{7}{3} & \frac{1}{3} & 1 & 0 \\ \frac{4}{3} & -\frac{4}{3} & 0 & 1 \\ 1 & 1 & -2 & 1 \\ 0 & 2 & 1 & -2 \end{bmatrix} \end{aligned} \quad (39)$$

is positive and asymptotically stable although the Metzler matrix (39) is not asymptotically stable.

#### 4. Concluding remarks

A method for decentralized stabilization of fractional positive descriptor continuous-time linear systems has been proposed. Necessary and sufficient conditions for decentralized stabilization of fractional positive descriptor systems have been established (Theorem 3). A procedure for computation of decentralized feedbacks has been proposed. The effectiveness of the procedure has been demonstrated on a numerical example. The considerations discussion can be easily extended to the fractional positive descriptor discrete-time linear systems and to fractional positive linear systems described by the new definitions of fractional derivatives (Caputo and Fabrizio, 2015; Losada and Nieto, 2015). An open problem is the extension of the method to positive nonlinear systems.

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