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DESCRIPTOR FRACTIONAL LINEAR SYSTEMS WITH REGULAR PENCILS

TADEUSZ KACZOREK

Faculty of Electrical Engineering
Białystok Technical University, ul. Wiejska 45D, 15-351 Białystok, Poland
e-mail: kaczorek@isep.pw.edu.pl

Methods for finding solutions of the state equations of descriptor fractional discrete-time and continuous-time linear systems with regular pencils are proposed. The derivation of the solution formulas is based on the application of the $\mathcal{Z}$ transform, the Laplace transform and the convolution theorems. Procedures for computation of the transition matrices are proposed. The efficiency of the proposed methods is demonstrated on simple numerical examples.

Keywords: descriptor system, fractional, system, regular pencil.

1. Introduction

Descriptor (singular) linear systems with regular pencils have been considered in many papers and books (Dodig and Stosic, 2009; Wang, 2012, Dai, 1989; Fahmy and O'Reill, 1989; Kaczorek, 2004; 1992; 2007a; 2007b; Kucera and Zagalak, 1988, Luenberger, 1978; Van Dooren, 1979). The eigenvalue and invariant assignment by state and output feedbacks was investigated by Dodig and Stosic (2009), Wang (2012), Dai (1989), Fahmy and O'Reill (1989) as well as Kaczorek (2004; 1992), while the realization problem for singular positive continuous-time systems with delays was discussed by Kaczorek (2007b). The computation of Kronecker's canonical form of a singular pencil was analyzed by Van Dooren (1979). A delay dependent criterion for a class of descriptor systems with delays varying in intervals was proposed by Wang (2012).

Fractional positive continuous-time linear systems were addressed by Kaczorek (2008), along with positive linear systems with different fractional order (Kaczorek, 2007a). A new concept of practical stability of positive fractional 2D systems was proposed by Kaczorek (2010b), who also presented an analysis of fractional linear electrical circuits (Kaczorek, 2012a) and some selected problems in the theory of fractional linear systems (Kaczorek, 2011b).

A new class of descriptor fractional linear systems and electrical circuits was introduced, their solution of state equations was derived and a method for decomposition of the descriptor fractional linear systems with regular pencils into dynamic and static parts was proposed by Kaczorek (2012a), who also considered positive fractional continuous-time linear systems with singular pencils (Kaczorek, 2012b). Fractional-order iterative learning control for fractional-order systems was addressed by Yan *et al.* (2011c).

In this paper, methods of finding solutions of the state equations of descriptor fractional discrete-time and continuous-time linear systems with regular pencils will be proposed.

The paper is organized as follows. In Section 2 the solution to the state equation of the descriptor system is derived using the method based on the $\mathcal{Z}$ transform and the convolution theorem. A method for computation of the transition matrix is proposed and illustrated on a simple numerical example in Section 3. In Section 4 the proposed method is extended to continuous-time linear systems. Concluding remarks are given in Section 5.

The following notation will be used: $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^{n\times m}$ is the set of real $n\times m$ matrices and $\mathbb{R}^n = \mathbb{R}^{n\times 1}$, $\mathcal{Z}_+$ is the set of nonnegative integers, I_n is the $n\times n$ identity matrix.

2. Discrete-time fractional linear systems

Consider the descriptor fractional discrete-time linear system

$$E\Delta^\alpha x_{i+1} = Ax_i + Bu_i, \qquad i\in\mathbb{Z}_+ = \{0,1,2,\dots\}, \quad 0<\alpha<1, \tag{1}$$

where α is the fractional order, $x_i \in \mathbb{R}^n$ is the state vector $u_i \in \mathbb{R}^m$ is the input vector and $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$. It is assumed that $\det E = 0$, but the pencil (E, A) is regular, i.e.,

$$\det[Ez - A] \neq 0 \quad \text{for some} z \in \mathbb{C}. \tag{2}$$

Without lost of generality we may assume that

$$E = \begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad E_1 \in \mathbb{R}^{r\times r}$$

and

$$\operatorname{rank} E_1 = \operatorname{rank} E = r < n. \tag{3}$$

Admissible boundary conditions for (1) are given by x_0. The fractional difference of the order $\alpha \in [0, 1)$ is defined by

$$\Delta^\alpha x_i = \sum_{k=0}^{i} c_k x_{i-k}, \tag{4a}$$

where

$$c_k = (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix}, \quad k = 0, 1, \dots \tag{4b}$$

and

$$\begin{pmatrix} \alpha \\ k \end{pmatrix} = \begin{cases} 1 & \text{for} \quad k = 0, \\ \dfrac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} & \text{for} \quad k = 1, 2, \dots \end{cases} \tag{4c}$$

Substitution of (4a) into (1) yields

$$Ex_{i+1} = Fx_i + \sum_{k=2}^{i+1} c_k x_{i-k+1} + Bu_i, \quad i \in \mathbb{Z}_+, \tag{5}$$

where $F = A - Ec_1 = A - E\alpha$.

Applying to (5) the $\mathcal{Z}$-transform and taking into account that (Kaczorek, 1992)

$$\begin{aligned} &\mathcal{Z}[x_{i-p}] \\ &= z^{-p}X(z) + z^{-p} \sum_{j=-1}^{-p} x_j z^{-j}, \quad p = 1, 2, \dots, \end{aligned} \tag{6}$$

we obtain

$$X(z) = [Ex - F]^{-1}\{Ex_0 z - H(z) + BU(z)\}, \tag{7a}$$

where

$$\begin{aligned} X(z) &= \mathcal{Z}[x_i] = \sum_{i=0}^{\infty} x_i z^{-i}, \\ U(z) &= \mathcal{Z}[u_i] = \sum_{i=0}^{\infty} u_i z^{-i}, \\ H(z) &= \mathcal{Z}[h_i], \quad h_i = \sum_{k=2}^{i+1} Ec_k x_{i-k+1}. \end{aligned} \tag{7b}$$

Let

$$[Ex - F]^{-1} = \sum_{j=-\mu}^{\infty} \psi_j z^{-(j+1)}, \tag{8}$$

where μ is the positive integer defined by the pair (E, A) (Kaczorek, 1992; Van Dooren, 1979). Comparison of the coefficients at the same powers of z of the equality

$$\begin{aligned} &[Ex - F] \left(\sum_{j=-\mu}^{\infty} \psi_j z^{-(j+1)} \right) \\ &= \left(\sum_{j=-\mu}^{\infty} \psi_j z^{-(j+1)} \right) [Ex - F] = I_n \end{aligned} \tag{9a}$$

yields

$$E\psi_{-\mu} = \psi_{-\mu} E = 0 \tag{9b}$$

and

$$\begin{aligned} &E\psi_k + E\psi_{k+1} \\ &= \psi_k E + \psi_{k-1} E \\ &= \begin{cases} I_n & \text{for} \quad k = 0\,, \\ 0 & \text{for} \quad k = 1-\mu, 2-\mu, \dots, -1, 1, 2, \dots \end{cases} \end{aligned} \tag{9c}$$

From (9b) and (9c) we have the matrix equation

$$G \begin{bmatrix} \psi_{0\mu} \\ \psi_{1N} \end{bmatrix} = \begin{bmatrix} V \\ 0 \end{bmatrix}, \tag{10a}$$

where

$$G = \begin{bmatrix} G_1 & 0 \\ G_{21} & G_2 \end{bmatrix} \in \mathbb{R}^{(N+\mu+1)n\times(N+\mu+1)n},$$

$$G_{21} = \begin{bmatrix} 0 & \dots & 0 & F \\ 0 & \dots & 0 & 0 \\ \vdots & \dots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{Nn\times(\mu+1)n},$$

$$G_1 = \begin{bmatrix} E & 0 & 0 & \dots & 0 & 0 & 0 \\ F & E & 0 & \dots & 0 & 0 & 0 \\ 0 & F & E & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & F & E & 0 \\ 0 & 0 & 0 & \dots & 0 & F & E \end{bmatrix} \in \mathbb{R}^{(\mu+1)n\times(\mu+1)n},$$

$$G_2 = \begin{bmatrix} E & 0 & 0 & \dots & 0 & 0 & 0 \\ F & E & 0 & \dots & 0 & 0 & 0 \\ 0 & F & E & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & F & E & 0 \\ 0 & 0 & 0 & \dots & 0 & F & E \end{bmatrix} \in \mathbb{R}^{Nn\times Nn},$$

$$\psi_{0\mu} = \begin{bmatrix} \psi_{-\mu} \\ \psi_{1-\mu} \\ \vdots \\ \psi_0 \end{bmatrix} \in \mathbb{R}^{(\mu+1)n\times n},$$

$$\psi_{1N} = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{bmatrix} \in \mathbb{R}^{Nn\times n},$$

$$V = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_n \end{bmatrix} \in \mathbb{R}^{(\mu+1)n\times n}. \tag{10b}$$

Equation (10a) has the solution

$$\begin{bmatrix} \psi_{0\mu} \\ \psi_{1N} \end{bmatrix}$$

for given G and V if and only if

$$\operatorname{rank}\left\{G, \begin{bmatrix} V \\ 0 \end{bmatrix}\right\} = \operatorname{rank} G. \tag{11}$$

It is easy to show that the condition (11) is satisfied if the condition (2) is met.

Substituting (8) into (7a) we obtain

$$X(z) = \left(\sum_{j=-\mu}^{\infty} \psi_j z^{-(j+1)}\right)[Ex_0 z - H(z) + BU(z)]. \tag{12}$$

Applying the inverse transform $\mathcal{Z}^{-1}$ and the convolution theorem to (12), we obtain

$$x_i = \psi_i E x_0 - \sum_{k=0}^{i+\mu-1} \psi_{i-k-1} \sum_{j=2}^{k+1} c_j x_{k-j+1} + \sum_{k=0}^{i+\mu-1} \psi_{i-k-1} B u_k. \tag{13}$$

To find the solution to Eqn. (1), first we compute the transition matrices ψ_j for $j = -\mu, 1-\mu, \dots, 1, 2, \dots$ and next, using (13), we obtain the desired solution.

3. Computation of transition matrices

To compute the transition matrices ψ_k for $k = -\mu, 1-\mu, \dots, N, \dots$, the following procedure is recommended.

Procedure 1.

Step 1. Find a solution $\psi_{0\mu}$ of the equation

$$G_1 \psi_{0\mu} = V, \tag{14}$$

where G_1, $\psi_{0\mu}$ and V are defined by (10b). Note that, if the matrix E has the form (3), then the first r rows of the matrix $\psi_{0\mu}$ are zero and its $n-r$ last rows are arbitrary.

Step 2. Choose $n-r$ arbitrary rows of the matrix ψ_0 so that

$$\operatorname{rank}\left\{\begin{bmatrix} E & 0 \\ F & E \end{bmatrix}, \begin{bmatrix} I_n - F\psi_{-1} \\ 0 \end{bmatrix}\right\} = \operatorname{rank}\begin{bmatrix} E & 0 \\ F & E \end{bmatrix} \tag{15}$$

and the equation

$$\begin{bmatrix} E & 0 \\ F & E \end{bmatrix}\begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix} = \begin{bmatrix} I_n - F\psi_{-1} \\ 0 \end{bmatrix}$$

has a solution with arbitrary last $n-r$ rows of the matrix ψ_1.

Step 3. Knowing $\psi_{0\mu}$, choose the last $n-r$ rows of the matrix ψ_1 so that

$$\operatorname{rank}\left\{\begin{bmatrix} E & 0 \\ F & E \end{bmatrix}, \begin{bmatrix} F\psi_0 \\ 0 \end{bmatrix}\right\} = \operatorname{rank}\begin{bmatrix} E & 0 \\ F & E \end{bmatrix} \tag{16}$$

and the equation

$$\begin{bmatrix} E & 0 \\ F & E \end{bmatrix}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = -\begin{bmatrix} F \\ 0 \end{bmatrix}\psi_0 \tag{17}$$

has a solution with arbitrary last $n-r$ rows of the matrix ψ_2. Repeating the last step for

$$\begin{bmatrix} \psi_2 \\ \psi_3 \end{bmatrix}, \begin{bmatrix} \psi_3 \\ \psi_4 \end{bmatrix}, \dots$$

we may compute the desired matrices ψ_k for $k = -\mu, 1-\mu, \dots$.

The details of the procedure will be shown on the following example.

Example 1. Find the solution to Eqn. (1) for $\alpha = 0.5$ with the matrices

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{18}$$

and the initial condition

$$x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and u_i, $\quad i \in \mathbb{Z}_+$.

In this case the pencil (2) of (18) is regular since

$$\det[Ez - A] = \begin{vmatrix} z & 0 \\ -1 & 2 \end{vmatrix} = 2z, \tag{19}$$

$\mu = 1$ and

$$F = [E\alpha - A] = \begin{bmatrix} \alpha & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ -1 & 2 \end{bmatrix}. \tag{20}$$

Using Procedure 1, we obtain the following.

Step 1. In this case Eqn. (14) has the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 1 & 0 \\ -1 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_{-1} \\ \psi_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{21}$$

and its solution with the arbitrary second row $[\ \psi_{21}^0 \quad \psi_{22}^0\ ]$ of ψ_0 is given by

$$\begin{bmatrix} \psi_{-1} \\ \psi_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \\ 1 & 0 \\ \psi_{21}^0 & \psi_{22}^0 \end{bmatrix}. \tag{22}$$

Step 2. We choose the row $[\ \psi_{21}^0 \quad \psi_{22}^0\ ]$ of ψ_0 so that (15) holds, i.e.,

$$\operatorname{rank} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha & 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 1 & 0 \\ -1 & 2 & 0 & 0 \end{bmatrix}, \tag{23}$$

and the equation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 1 & 0 \\ -1 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{24}$$

has the solution

$$\begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.5 & 0 \\ -\alpha & 0 \\ \psi_{21}^1 & \psi_{22}^1 \end{bmatrix} \tag{25}$$

with the second arbitrary row $[\ \psi_{21}^1 \quad \psi_{22}^1\ ]$ of ψ_1.

Step 3. We choose $[\ \psi_{21}^1 \quad \psi_{22}^1\ ]$ so that the equation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 1 & 0 \\ -1 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\alpha & 0 \\ \psi_{21}^1 & \psi_{22}^1 \\ \alpha^2 & 0 \\ \psi_{21}^2 & \psi_{22}^2 \end{bmatrix} = -\begin{bmatrix} \alpha & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{26}$$

has the solution

$$\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} -\alpha & 0 \\ -0.5\alpha & 0 \\ \alpha^2 & 0 \\ \psi_{21}^2 & \psi_{22}^2 \end{bmatrix} \tag{27}$$

with arbitrary $[\ \psi_{21}^2 \quad \psi_{22}^2\ ]$.

Continuing the procedure, we obtain

$$\psi_{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad \psi_k = (-1)^k \begin{bmatrix} \alpha^k & 0 \\ 0.5\alpha^k & 0 \end{bmatrix} \tag{28}$$

for $k = 0, 1, \dots$

Using (13), (18) and (20), we obtain the desired solution of the form

$$x_i = \psi_i \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sum_{k=0}^{i} \psi_{i-k-1} \sum_{j=2}^{k+1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} c_j x_{k-j+1} + \sum_{k=0}^{i} \psi_{i-k-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} u_k, \tag{29}$$

where c_j are defined by (4b). ♦

4. Continuous-time fractional linear systems

Consider the descriptor fractional continuous-time linear system described by the state equations

$$ED^\alpha x(t) = Ax(t) + Bu(t), \quad n-1 < \alpha \le n \in \{1, 2, \dots\} \tag{30a}$$

$$y(t) = Cx(t) + Du(t), \tag{30b}$$

where D^α is the Caputo differentiation operator, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the state, input and output vectors and $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

It is assumed that $\det E = 0$ and the pencil (E, A) is regular, i.e.,

$$\det[E\lambda - A] \ne 0 \quad \text{for some} \quad \lambda \in \mathcal{C}. \tag{31}$$

Admissible initial conditions for (30a) are given by

$$x^{(k)}(0) = x_{k,0} \quad \text{for} \quad k = 0, 1, \dots, n-1. \tag{32}$$

Applying the Laplace transform $(\mathcal{L})$ to Eqn. (30a), we obtain (Kaczorek, 2011b)

$$[Es^\alpha - A]X(s) = BU(s) + \sum_{k=0}^{n-1} s^{\alpha-k-1} x_{k,0}, \tag{33}$$

where $X(s) = \mathcal{L}[x(t)]$ and $U(s) = \mathcal{L}[u(t)]$. If the condition (31) is satisfied, then from (33) we obtain

$$X(s) = [Es^\alpha - A]^{-1} \left(BU(s) + \sum_{k=0}^{n-1} s^{\alpha-k-1} x_{k,0} \right). \tag{34}$$

In a particular case when $\det E \neq 0$, from (34) we have

$$X(s) = \sum_{i=0}^{\infty}(E^{-1}A)^i E^{-1} s^{-(i+1)\alpha} \times \left(BU(s) + \sum_{k=0}^{n-1} s^{\alpha-k-1} x_{k,0} \right) \quad (35)$$

since

$$[Es^\alpha - A]^{-1} = [Es^\alpha(I_n - (Es^\alpha)^{-1})A]^{-1} = \sum_{i=0}^{\infty}(E^{-1}A)^i E^{-1} s^{-(i+1)\alpha}.$$

Using the inverse Laplace transform ($\mathcal{L}^{-1}$) and the convolution theorem, we obtain

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}[X(s)] \\ &= \sum_{i=0}^{\infty} \Big[\int_0^t (E^{-1}A)^i \frac{(t-\tau)^{(i+1)\alpha-1}}{\Gamma[(i+1)\alpha]} E^{-1} Bu(\tau)\,\mathrm{d}\tau \\ &\quad + \sum_{k=0}^{n-1} \frac{t^{i\alpha+k}}{\Gamma(i\alpha+k+1)} (E^{-1}A)^i x_{k,0} \Big], \end{aligned} \quad (36)$$

where $\Gamma(\alpha)$ is the gamma function (Kaczorek, 2011b).

Therefore, the following theorem for $\det E \neq 0$ has been proved.

Theorem 1. *The solution of Eqn. (30a) for* $\det E \neq 0$ *and the initial conditions (32) is given by (36).*

If $E = I_n$*, then (36) takes the form (Kaczorek, 2011b)*

$$x(t) = \sum_{l=1}^{n} \Phi_l(t) x^{(l-1)}(0^+) + \int_0^t \Phi(t-\tau) Bu(\tau)\,\mathrm{d}\tau, \quad (37)$$

where

$$\Phi_l(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k\alpha+l)-1}}{\Gamma(k\alpha+l)}, \quad (38a)$$

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}. \quad (38b)$$

If $\det E = 0$ but the pencil is regular (the condition (31) is met), then

$$[Es^\alpha - A]^{-1} = \sum_{i=-\mu}^{\infty} T_i s^{-(i+1)\alpha}, \quad (39)$$

where T_i satisfy the equality

$$ET_i - AT_{i-1} = T_i E - T_{i-1}A = \begin{cases} I_n & \text{for } i = 0, \\ 0 & \text{for } i \neq 0, \end{cases} \quad (40)$$

and $T_i = 0$ for $i < -\mu, T_{-\mu}E = ET_{-\mu} = 0$.

The equality (40) follows from the comparison of the coefficients at the same powers of *s* in the equality

$$[Es^\alpha - A]\left(\sum_{i=-\mu}^{\infty} T_i s^{-(i+1)\alpha}\right) = \left(\sum_{i=-\mu}^{\infty} T_i s^{-(i+1)\alpha}\right)[Es^\alpha - A] = I_n.$$

Substitution of (39) into (34) yields

$$\begin{aligned} X(s) &= \sum_{i=-\mu}^{\infty} T_i \left[s^{-(i+1)\alpha} BU(s) + \sum_{k=0}^{n-1} s^{-i\alpha-k-1} x_{k,0} \right] \\ &= \sum_{i=0}^{\infty} T_i \left[s^{-(i+1)\alpha} BU(s) + \sum_{k=0}^{n-1} s^{-i\alpha-k-1} x_{k,0} \right] \\ &\quad + \sum_{i=1}^{\mu} T_{-i} \left[s^{(i+1)\alpha} BU(s) + \sum_{k=0}^{n-1} s^{i\alpha-k-1} x_{k,0} \right]. \end{aligned} \quad (41)$$

Applying the inverse Laplace transform and the convolution theorem to (41), we obtain

$$\begin{aligned} x(t) &= \sum_{i=0}^{\infty} \Big[\int_0^t T_i B \frac{(t-\tau)^{(i+1)\alpha-1}}{\Gamma[(i+1)\alpha]} Bu(\tau)\,\mathrm{d}\tau \\ &\quad + \sum_{k=0}^{n-1} T_i \frac{t^{i\alpha+k}}{\Gamma(i\alpha+k+1)} Ex_{k,0} \Big] \\ &\quad + \sum_{i=1}^{\mu} T_{-i} \left[Bu(t)^{(i-1)\alpha} + \sum_{k=0}^{n-1} \delta^{(i\alpha-1)} Ex_{k,0} \right] \end{aligned} \quad (42)$$

or

$$\begin{aligned} x(t) &= \sum_{i=0}^{\infty} \Big[\int_0^t (T_0 A)^i T_0 B \frac{(t-\tau)^{(i+1)\alpha-1}}{\Gamma[(i+1)\alpha]} u(\tau)\,\mathrm{d}\tau \\ &\quad + \sum_{k=0}^{n-1} (T_0 A)^i T_0 \frac{t^{i\alpha+k}}{\Gamma(i\alpha+k+1)} x_{k,0} \Big] \\ &\quad + \sum_{i=1}^{\mu} T_{-i} \left[Bu(t)^{(i-1)\alpha} + \sum_{k=0}^{n-1} \delta^{(i\alpha-1)} Ex_{k,0} \right], \end{aligned} \quad (43)$$

since by (40) $T_i = (T_0 A)^i T_0$ for $i = 0, 1, \ldots$ and $\delta^{(k)}$ is the k-th derivative of the delta impulse function δ.

Therefore, the following theorem has been proved.

Theorem 2. *If the condition (31) is satisfied, then the solution of Eqn. (30a) with the admissible initial conditions (32) is given by (42) or (43).*

In a particular case of $0 < \alpha \le 1$, *from (43) we have*

$$x(t) = \sum_{i=0}^{\infty}(T_0A)^iT_0\Big[\int_0^t B\frac{(t-\tau)^{(i+1)\alpha-1}}{\Gamma[(i+1)\alpha]}u(\tau)\,\mathrm{d}\tau + \frac{t^{i\alpha}}{\Gamma(i\alpha+1)}Ex_0\Big] + \sum_{i=1}^{\mu}T_{-i}[Bu(t)^{(i-1)\alpha} + \delta^{(i\alpha-1)}Ex_0]. \tag{44}$$

To compute the matrices T_i *for* $i = -\mu, 1-\mu, \ldots,$ *the procedure given in Section 3 is recommended.*

Example 2. Consider Eqn. (30a) for $\alpha = 0.5$ with the matrices

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{45}$$

and the zero initial condition $x_0 = 0$. The pencil is regular since

$$\det[E\lambda - A] = \begin{vmatrix} \lambda & 0 \\ -1 & 2 \end{vmatrix} = 2\lambda, \quad (\lambda = s^\alpha) \tag{46}$$

and

$$[E\lambda - A]^{-1} = \begin{bmatrix} \lambda^{-1} & 0 \\ 0.5\lambda^{-1} & 0.5 \end{bmatrix} = T_{-1} + T_0\lambda^{-1}, \tag{47a}$$

where

$$T_{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad T_0 = \begin{bmatrix} 1 & 0 \\ 0.5 & 0 \end{bmatrix}. \tag{47b}$$

Using (42), (43) and (47b) we obtain

$$\begin{aligned} x(t) &= T_0B\int_0^t \frac{(t-\tau)^{-0.5}}{\Gamma(0.5)}u(\tau)\,\mathrm{d}\tau + T_{-1}Bu(t) \\ &= \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}\int_0^t \frac{(t-\tau)^{-0.5}}{\Gamma(0.5)}u(\tau)\,\mathrm{d}\tau + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t). \end{aligned} \tag{48}$$

♦

5. Concluding remarks

New methods of finding solutions of the state equations of descriptor fractional discrete-time and continuous-time linear systems with regular pencils have been proposed. Derivation of the solution formulas has been based on the application of the $\mathcal{Z}$-transform, the Laplace transform and the convolution theorems. A procedure for computation of the transition matrices has been proposed and its application has been demonstrated on simple numerical examples. An open problem is the extension of the method for 2D descriptor fractional discrete and continuous-discrete linear systems.

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Tadeusz Kaczorek received the M.Sc., Ph.D. and D.Sc. degrees in electrical engineering from the Warsaw University of Technology in 1956, 1962 and 1964, respectively. In the years 1968–69 he was the dean of the Electrical Engineering Faculty, and in the period of 1970–73 he was a deputy rector of the Warsaw University of Technology. In 1971 he became a professor and in 1974 a full professor at the same university. Since 2003 he has been a professor at Białystok Technical University. In 1986 he was elected a corresponding member and in 1996 a full member of the Polish Academy of Sciences. In the years 1988–1991 he was the director of the Research Centre of the Polish Academy of Sciences in Rome. In 2004 he was elected an honorary member of the Hungarian Academy of Sciences. He has been granted honorary doctorates by nine universities. His research interests cover systems theory, especially singular multidimensional systems, positive multidimensional systems, singular positive 1D and 2D systems, as well as positive fractional 1D and 2D systems. He initiated research in the field of singular 2D, positive 2D and positive fractional linear systems. He has published 24 books (six in English) and over 1000 scientific papers. He has also supervised 69 Ph.D. theses. He is the editor-in-chief of the *Bulletin of the Polish Academy of Sciences: Technical Sciences* and a member of editorial boards of ten international journals.

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