

# A MODIFIED STATE VARIABLE DIAGRAM METHOD FOR DETERMINATION OF POSITIVE REALIZATIONS OF LINEAR CONTINUOUS-TIME SYSTEMS WITH DELAYS

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A new modified state variable diagram method is proposed for determination of positive realizations of linear continuous-time systems with delays in state and input vectors. Using the method, it is possible to find a positive realization with reduced numbers of delays for a given transfer matrix. Sufficient conditions for the existence of positive realizations of given proper transfer matrices are established. The proposed method is demonstrated on numerical examples.

**Keywords:** state diagram method, determination, linear, continuous-time, delay, realization.

## 1. Introduction

Determination of the state space equations for a given transfer matrix is a classical problem, called the realization problem, which has been addressed in many papers and books (Farina and Rinaldi, 2000; Benvenuti and Farina, 2004; Kaczorek, 1992; 2008; 2011d; 2012a; 2012b; Shaker and Dixon, 1977). It is well known (Farina and Rinaldi, 2000; Kaczorek, 2002; 1992) that to find a realization for a given transfer function first we have to find a state matrix for given denominator of the transfer function.

Some overviews on the positive realization problem are given by Farina and Rinaldi (2000), Kaczorek (2002), as well as Benvenuti and Farina (2004). The realization problem for positive continuous-time and discrete-time linear systems was considered by Kaczorek (2006a; 2006b; 2011a; 2011b; 2006c; 2004; 2011c) along with the positive realization problem for discrete-time systems with delays (Kaczorek, 2006; 2004; 2005). Fractional positive linear systems were addressed by Kaczorek (2008c; 2009a; 2011d), together with the realization problem for fractional linear systems (Kaczorek, 2008a) and for positive 2D hybrid systems (Kaczorek, 2008b). A method based on a similarity transformation of the standard realization to the discrete positive one was proposed by Kaczorek (2011c). Conditions for the existence of a positive stable realization with a system Metzler matrix for transfer function were established by Kaczorek (2011a),

who also formulated and solved the problem of determination of the set of Metzler matrices for given stable polynomials (Kaczorek, 2012a).

In this paper a new modified state variable diagram method for determination of positive realizations with a reduced number of delays for given proper transfer matrices will be proposed and sufficient conditions for the existence of positive realizations will be established. The proposed method allows us to find a positive realization with the number of delays less than the one that follows from the degree of the denominator of a given transfer function.

The paper is organized as follows. In Section 2 some preliminaries concerning positive continuous-time linear systems with delays are recalled and the problem formulation is given. Basic lemmas of the proposed method are given in Section 3. The new modified state variable diagram method is proposed in Section 4. Concluding remarks are given in Section 5.

The following notation will be used:  $\mathbb{R}$  is the set of real numbers,  $\mathbb{R}^{n \times m}$  is the set of  $n \times m$  real matrices,  $\mathbb{R}_+^{n \times m}$  is the set of  $n \times m$  matrices with nonnegative entries and  $\mathbb{R}_+^n = \mathbb{R}_+^{n \times 1}$ ,  $M_n$  is the set of  $n \times n$  Metzler matrices (real matrices with nonnegative off-diagonal entries),  $I_n$  is the  $n \times n$  identity matrix,  $A^T$  is the transpose of a matrix  $A$ ,  $\mathbb{R}^{n \times m}(s, w)$  is the set of  $n \times m$  rational matrices in  $s$  and  $w$ ,  $\mathbb{R}^{n \times m}[s, w]$  is the set of  $n \times m$  polynomial matrices in  $s$  and  $w$ .

## 2. Preliminaries and problem formulation

Consider the continuous-time linear system with  $h$  delays in state and  $q$  delays in inputs,

$$\dot{x}(t) = \sum_{i=0}^h A_i x(t - id) + \sum_{j=0}^q B_j u(t - jd), \quad (1a)$$

$$y(t) = Cx(t) + Du(t), \quad (1b)$$

where  $x(t) \in \mathbb{R}^n$ ,  $u(t) \in \mathbb{R}^m$ ,  $y(t) \in \mathbb{R}^p$  are the state, input and output vectors, respectively, and  $A_i \in \mathbb{R}^{n \times n}$ ,  $i = 0, 1, \dots, h$ ,  $B_j \in \mathbb{R}^{n \times m}$ ,  $j = 0, 1, \dots, q$ ,  $C \in \mathbb{R}^{p \times n}$ ,  $D \in \mathbb{R}^{p \times m}$ ,  $d > 0$  being a delay.

Initial conditions for (1) are given by

$$\begin{aligned} x_0(t) & \text{ for } t \in [-hd, 0], \\ u_0(t) & \text{ for } t \in [-qd, 0]. \end{aligned} \quad (2)$$

**Definition 1.** The system (1) is called (internally) *positive* for every  $x_0(t) \in \mathbb{R}_+^n$ ,  $t \in [-hd, 0]$  if  $u_0(t) \in \mathbb{R}_+^m$ ,  $t \in [-qd, 0]$  and all inputs  $u(t) \in \mathbb{R}_+^m$ ,  $t \geq 0$  we have  $x(t) \in \mathbb{R}_+^n$  and  $y(t) \in \mathbb{R}_+^p$  for  $t \geq 0$ .

**Theorem 1.** (Kaczorek, 2005) *The system (1) is positive if and only if*

$$\begin{aligned} A_0 & \in M_n, A_i \in \mathbb{R}_+^{n \times n}, \quad i = 1, 2, \dots, h, \\ B_j & \in \mathbb{R}_+^{n \times m}, \quad j = 0, 1, \dots, q, \\ C & \in \mathbb{R}_+^{p \times n}, \quad D \in \mathbb{R}_+^{p \times m}. \end{aligned} \quad (3)$$

The transfer matrix of the system (1) is given by

$$\begin{aligned} T(s, w) &= C[I_n s - A_0 - A_1 w - \dots - A_h w^h]^{-1} \\ & \quad [B_0 + B_1 w + \dots + B_q w^q] + D, \end{aligned} \quad (4)$$

$$w = e^{-ds}.$$

**Definition 2.** The matrices (3) are called a *positive realization* of a given transfer matrix  $T(s, w) \in \mathbb{R}^{p \times m}(s, w)$  if they satisfy (4).

The positive realization problem under consideration can be stated as follows: Given a proper transfer matrix  $T(s, w) \in \mathbb{R}^{p \times m}(s, w)$ , find a positive realization with reduced numbers of delays (3) of  $T(s, w)$ .

In this paper sufficient conditions for the problem solvability will be established and a new method for determination of a positive realization with a reduced number of delays will be proposed.

## 3. Problem solution

The transfer matrix (4) can be written in the following form:

$$\begin{aligned} T(s, w) &= \frac{C(H_{ad}(s, w))[B_0 + B_1 w + \dots + B_q w^q]}{\det H(s, w)} + D \\ &= \frac{N(s, w)}{d(s, w)} + D, \end{aligned} \quad (5)$$

where

$$\begin{aligned} H(s, w) &= [I_n s - A_0 - A_1 w - \dots - A_h w^h] \in \mathbb{R}^{n \times n}[s, w], \end{aligned} \quad (6)$$

$$\begin{aligned} N(s, w) &= C(H_{ad}(s, w))[B_0 + B_1 w + \dots + B_q w^q], \\ d(s, w) &= \det H(s, w). \end{aligned} \quad (7)$$

From (5) we have

$$D = \lim_{s \rightarrow \infty} T(s, w) \quad (8)$$

since  $\lim_{s \rightarrow \infty} H^{-1}(s, w) = 0$ . The strictly proper transfer matrix is given by

$$T_{sp}(s, w) = T(s, w) - D = \frac{N(s, w)}{d(s, w)}. \quad (9)$$

Therefore, the positive realization problem is reduced to finding the matrices

$$\begin{aligned} A_0 & \in M_n, \quad A_i \in \mathbb{R}_+^{n \times n}, \quad i = 1, 2, \dots, h, \\ B_j & \in \mathbb{R}_+^{n \times m}, \quad j = 0, 1, \dots, q, \quad C \in \mathbb{R}_+^{p \times n}, \end{aligned} \quad (10)$$

for a given strictly proper transfer matrix (9).

To simplify the notation, we shall consider a Single-Input Single-Output (SISO) system described by (1) for  $m = p = 1$ .

Let a given, strictly proper, irreducible transfer function have the form

$$T_{sp}(s, w) = \frac{n(s, w)}{d(s, w)}, \quad (11a)$$

$$\begin{aligned} n(s, w) &= b_{n-1}(w)s^{n-1} + \dots + b_1(w)s + b_0(w), \\ b_k(w) &= b_{k,m}w^m + \dots + b_{k,1}w + b_{k,0}, \\ k &= 0, 1, \dots, n-1, \end{aligned} \quad (11b)$$

$$\begin{aligned} d(s, w) &= s^n - a_{n-1}(w)s^{n-1} - \dots \\ & \quad - a_1(w)s - a_0(w), \\ a_k(w) &= a_{k,m}w^m + \dots + a_{k,1}w + a_{k,0}, \\ k &= 0, 1, \dots, n-1. \end{aligned} \quad (11c)$$

The solution of the positive realization problem for (11) is based on the following two lemmas (Kaczorek, 2006a).

**Lemma 1.** *Let  $p_k = p_k(w)$  for  $k = 1, 2, \dots, 2n-1$  be some polynomials in  $w$  with nonnegative coefficients and*

$$P(w) = \begin{bmatrix} 0 & 0 & \dots & 0 & p_n \\ p_1 & 0 & \dots & 0 & p_{n+1} \\ 0 & p_2 & \dots & 0 & p_{n+2} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & p_{n-1} & p_{2n-1} \end{bmatrix}. \quad (12)$$

Then

$$\begin{aligned} \det[I_n s - P(w)] \\ = s^n - p_{2n-1}s^{n-1} - p_{n-1}p_{2n-2}s^{n-2} - \dots \\ - p_2p_3 \dots p_{n-1}p_{n+1}s - p_1p_2 \dots p_n. \end{aligned} \quad (13)$$

The proof is given by Kaczorek (2006a).

**Lemma 2.** Let  $R_n(w)$  be the  $n$ -th row of the adjoint matrix  $[I_n s - P(w)]_{ad}$ . Then

$$\begin{aligned} R_n(w) \\ = [p_1p_2 \dots p_{n-1} \quad p_2p_3 \dots p_{n-1}s \quad p_3p_4 \dots p_{n-1}s^2 \\ \dots p_{n-1}s^{n-2} \quad s^{n-1}]. \end{aligned} \quad (14)$$

The proof is given by Kaczorek (2006a).

From Lemmas 1 and 2 we obtain that, if

$$P(w) = \begin{bmatrix} 0 & 0 & \dots & 0 & p_2 \\ p_1 & 0 & \dots & 0 & p_3 \\ 0 & p_1 & \dots & 0 & p_4 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & p_1 & p_{n+1} \end{bmatrix}, \quad (15)$$

then

$$\begin{aligned} \det[I_n s - P(w)] \\ = s^n - p_{n+1}s^{n-1} - \dots - p_3p_1^{n-2}s - p_2p_1^{n-1}, \end{aligned} \quad (16)$$

and

$$R_n(s) = [p_1^{n-1} \quad p_1^{n-2}s \quad \dots \quad p_1s^{n-2} \quad s^{n-1}]. \quad (17)$$

It is assumed that for a given denominator (11c) there exist polynomials

$$\begin{aligned} p_k = p_k(w) = p_{k,h}w^h + \dots + p_{k,1}w + p_{k,0}, \\ k = 0, 1, \dots, 2n-1, \end{aligned} \quad (18)$$

with nonnegative coefficients  $p_{k,j}$ ,  $j = 0, 1, \dots, h$  such that

$$\begin{aligned} a_{n-1}(w) &= p_{2n-1}, \\ a_{n-2}(w) &= p_{n-1}p_{2n-2}, \dots, \\ a_1(w) &= p_2p_3 \dots p_{n-1}p_{n+1}, \\ a_0(w) &= p_1p_2 \dots p_n. \end{aligned} \quad (19)$$

In a particular case, if the matrix  $P(w)$  has the form (15), then (19) takes the form

$$a_k(w) = p_1^{n-k-1}p_{k+2}, \quad k = 0, 1, \dots, n-1. \quad (20)$$

Note that if the assumption (19) is satisfied, then for a given denominator  $d, s, w$  of (11a) we may find the matrix (12) and next the corresponding matrices  $A_i \in \mathbb{R}_+^{n \times n}$ ,  $i = 1, 2, \dots, h$ , since

$$I_n s - P(w) = I_n s - \sum_{i=0}^h A_i w^i. \quad (21)$$

The matrix  $C$  is chosen in the form

$$C = [0 \quad \dots \quad 0 \quad 1] \in \mathbb{R}^{1 \times n}. \quad (22)$$

Taking into account (14), (21) and (7), we obtain

$$\begin{aligned} C[I_n s - P(w)]_{ad}[B_0 + B_1 w + \dots + B_q w^q] \\ = R_n(w)[B_0 + B_1 w + \dots + B_q w^q] \\ = [p_1p_2 \dots p_{n-1} \quad p_2p_3 \dots p_{n-1}s \quad p_3p_4 \dots p_{n-1}s^2 \\ \dots p_{n-1}s^{n-2} \quad s^{n-1}] \\ [B_0 + B_1 w + \dots + B_q w^q] \\ = n(s, w). \end{aligned} \quad (23)$$

#### 4. Modified state variables diagram method

First the modified state variables diagram method of determination of positive realizations is presented on the strictly proper transfer function (24).

The proposed method is based on Lemmas 1 and 2. It is assumed that there exist polynomials (18) with nonnegative coefficients  $p_{k,j}$ ,  $j = 0, 1, \dots, h$ , (except the last coefficient  $p_{k,0}$ ) satisfying (19) and

$$\begin{aligned} b_{n-1}(w) &= \bar{b}_{n-1}(w), \\ b_{n-2}(w) &= p_{n-1}\bar{b}_{n-2}(w), \\ &\vdots \\ b_1(w) &= p_2p_3 \dots p_{n-1}\bar{b}_1(w), \\ b_0(w) &= p_1p_2 \dots p_{n-1}\bar{b}_0(w), \end{aligned} \quad (25)$$

for some polynomials with nonnegative coefficients  $\bar{b}_{n-2}(w), \dots, \bar{b}_1(w), \bar{b}_0(w)$ .

For (24) we have

$$\begin{aligned} p_1(w) &= w^2, \quad p_2(w) = w + 1, \quad p_3(w) = w^2 + w + 2, \\ p_4(w) &= w^2 + 2w, \quad p_5(w) = 2w^2 + 3w - 1, \end{aligned} \quad (26)$$

since

$$\begin{aligned} \det[I_3 s - P(w)] \\ = \begin{vmatrix} s & 0 & -w^2 - w - 2 \\ -w^2 & s & -w^2 - 2w \\ 0 & -w - 1 & s - 2w^2 - 3w + 1 \end{vmatrix} \\ = s^3 - (2w^2 + 3w - 1)s^2 \\ - (w^3 + 3w^2 + 2w)s \\ - (w^5 + 2w^4 + 3w^3 + 2w^2) \\ = s^3 - a_2(w)s^2 - a_1(w)s - a_0(w), \\ a_2(w) &= p_5(w), \quad a_1(w) = p_2(w)p_4(w), \\ a_0(w) &= p_1(w)p_2(w)p_3(w), \end{aligned} \quad (27a)$$

$$\begin{aligned}
 T_{sp}(s, w) &= \frac{b_2(w)s^2 + b_1(w)s + b_0(w)}{s^3 - a_2(w)s^2 - a_1(w)s - a_0(w)} \\
 &= \frac{(3w^2 + w + 2)s^2 + (w^2 + 3w + 2)s + w^4 + 2w^3 + w^2}{s^3 - (2w^2 + 3w - 1)s^2 - (w^3 + 3w^2 + 2w)s - (w^5 + 2w^4 + 3w^3 + 2w^2)}.
 \end{aligned} \quad (24)$$

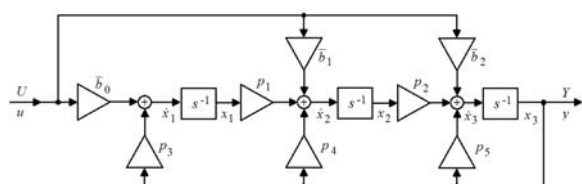


Fig. 1. State variable diagram for (28).

and

$$\begin{aligned}
 b_2(w) &= 3w^2 + w + 2, \quad b_1(w) = p_2(w), \\
 \bar{b}_1(w) &= (w + 1)(w + 2) = w^2 + 3w + 2, \\
 b_0(w) &= p_1(w)p_2(w)\bar{b}_0(w) \\
 &= (w^3 + w^2)(w + 1) = w^4 + 2w^3 + w^2, \quad (27b)
 \end{aligned}$$

where

$$\bar{b}_1(w) = w + 2, \quad \bar{b}_0(w) = w + 1. \quad (27c)$$

From (24) written in the form

$$\frac{b_2(w)s^{-1} + b_1(w)s^{-2} + b_0(w)s^{-3}}{1 - a_2(w)s^{-1} - a_1(w)s^{-2} - a_0(w)s^{-3}} = \frac{Y}{U},$$

we have

$$\begin{aligned}
 Y &= s^{-1}\{b_2(w)U + a_2(w)Y + s^{-1}[b_1(w)U \\
 &\quad + a_1(w)Y + s^{-1}(b_0(w)U + a_0(w)Y)]\} \\
 &= s^{-1}\{\bar{b}_2(w)U + p_5(w)Y + s^{-1}[p_2(w)\bar{b}_1(w)U \\
 &\quad + p_2(w)p_4(w)Y + s^{-1}(p_1(w)p_2(w)\bar{b}_0(w)U \\
 &\quad + p_1(w)p_2(w)p_3(w)Y)]\}.
 \end{aligned} \quad (28)$$

Using (28) we may draw the modified state variable diagram shown in Fig. 1. The variables  $x_1, x_2, x_3$  are chosen as the outputs of the integral elements. Using the modified state variables diagram, we can write the following equations:

$$\begin{aligned}
 \dot{x}_1 &= p_3x_3 + \bar{b}_0u, \\
 \dot{x}_2 &= p_1x_1 + p_4x_3 + \bar{b}_1u, \\
 \dot{x}_3 &= p_2x_2 + p_5x_3 + \bar{b}_2u,
 \end{aligned} \quad (29a)$$

which can be also written in the form

$$\dot{x} = Ax + Bu, \quad (29b)$$

where

$$\begin{aligned}
 A &= \begin{bmatrix} 0 & 0 & p_3 \\ p_1 & 0 & p_4 \\ 0 & p_2 & p_5 \end{bmatrix} = P(w) = A_0 + A_1w + A_2w^2, \\
 A_0 &= \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}, \\
 A_2 &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} \bar{b}_0 \\ \bar{b}_1 \\ \bar{b}_2 \end{bmatrix}.
 \end{aligned} \quad (29c)$$

Assuming  $C = [0 \ 0 \ 1]$  and taking into account (29b), we have

$$\begin{aligned}
 T_{sp}(s, w) &= C[I_3s - A]^{-1}B \\
 &= [0 \ 0 \ 1] \begin{bmatrix} s & 0 & -p_3 \\ -p_1 & s & -p_4 \\ 0 & -p_2 & s - p_5 \end{bmatrix}^{-1} \begin{bmatrix} \bar{b}_0 \\ \bar{b}_1 \\ \bar{b}_2 \end{bmatrix} \\
 &= \frac{[p_1p_2 \quad p_2s \quad s^2]}{s^3 - p_5s^2 - p_2p_4s - p_1p_2p_3} \begin{bmatrix} \bar{b}_0 \\ \bar{b}_1 \\ \bar{b}_2 \end{bmatrix} \\
 &= \frac{p_1p_2\bar{b}_0 + p_2\bar{b}_1s + \bar{b}_2s^2}{s^3 - a_2(w)s^2 - a_1(w)s - a_0(w)}.
 \end{aligned} \quad (30)$$

Comparison of (30) and (24) yields

$$\begin{aligned}
 \bar{b}_2(w) &= b_2(w) = 3w^2 + w + 2, \\
 \bar{b}_1(w) &= w + 2, \quad \bar{b}_0(w) = w + 1
 \end{aligned} \quad (31a)$$

and

$$\bar{B} = \begin{bmatrix} \bar{b}_0(w) \\ \bar{b}_1(w) \\ \bar{b}_2(w) \end{bmatrix} = B_0 + B_1w + B_2w^2, \quad (31b)$$

where

$$B_0 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}. \quad (31c)$$

The desired positive realization of (24) is given by (29c), (31c) and  $C = [0 \ 0 \ 1]$ ,  $D = [0]$ .

Note that the sufficient condition for the existence of a positive realization of (24) is the existence of the polynomials (26) and  $\bar{b}_1(w), \bar{b}_0(w)$  with nonnegative coefficients satisfying (27a) and (27b).

In a general case, consider the strictly proper irreducible transfer function

$$T_{sp}(s, w) = \frac{n(s, w)}{d(s, w)}, \quad (32a)$$

where

$$\begin{aligned} n(s, w) &= b_{n-1}(w)s^{n-1} + \dots + b_1(w)s + b_0(w), \\ b_k(w) &= b_{k,m}w^m + \dots + b_{k,1}w + b_{k,0}, \\ k &= 0, 1, \dots, n-1, \end{aligned} \quad (32b)$$

$$\begin{aligned} d(s, w) &= s^n - a_{n-1}(w)s^{n-1} - \dots - a_1(w)s - a_0(w), \\ a_k(w) &= a_{k,m}w^m + \dots + a_{k,1}w + a_{k,0}, \\ k &= 0, 1, \dots, n-1. \end{aligned} \quad (32c)$$

It is assumed that there exist the polynomials (32) and  $\bar{b}_0(w), \bar{b}_1(w), \dots, \bar{b}_{n-2}(w)$  with nonnegative coefficients (except the last coefficient of  $p_{2n-1}(w)$  satisfying (25).

By Lemma 1 we have

$$\begin{aligned} \det[I_n s - P(w)] &= \begin{vmatrix} s & 0 & \dots & 0 & -p_n \\ -p_1 & s & \dots & 0 & -p_{n+1} \\ 0 & -p_2 & \dots & 0 & -p_{n+2} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & -p_{n-1} & s - p_{2n-1} \end{vmatrix} \\ &= s^n - p_{2n-1}s^{n-1} - p_{n-1}p_{2n-2}s^{n-2} - \dots \\ &\quad - p_2p_3 \dots p_{n-1}p_{n+1}s - p_1p_2 \dots p_n, \end{aligned} \quad (33)$$

and, by Lemma 2, the  $n$ -th row of  $R_n(w)$  of the adjoint matrix  $[I_n s - P(w)]_{ad}$  is

$$\begin{aligned} R_n(w) &= [p_1p_2 \dots p_{n-1} \quad p_2p_3 \dots p_{n-1}s \quad p_3p_4 \dots p_{n-1}s^2 \\ &\quad \dots \quad p_{n-1}s^{n-2} \quad s^{n-1}]. \end{aligned} \quad (34)$$

Let

$$\bar{B}(w) = \begin{bmatrix} \bar{b}_0(w) \\ \vdots \\ \bar{b}_{n-1}(w) \end{bmatrix} = \begin{bmatrix} \bar{b}_0 \\ \vdots \\ \bar{b}_{n-1} \end{bmatrix} \quad (35)$$

and

$$\begin{aligned} R_n(w)\bar{B}(w) &= p_1p_2 \dots p_{n-1}\bar{b}_0 + p_2p_3 \dots p_{n-1}\bar{b}_1s + \dots \\ &\quad + p_{n-1}\bar{b}_{n-2}s^{n-2} + \bar{b}_{n-1}s^{n-1} \\ &= b_0 + b_1s + \dots + b_{n-2}s^{n-2} + b_{n-1}s^{n-1} \\ &= n(s, w). \end{aligned} \quad (36)$$

Assuming  $C = [0 \dots 0 \ 1] \in \mathbb{R}_+^{1 \times n}$  and using (33), (36), we obtain

$$\begin{aligned} T_{sp}(s, w) &= C[I_n s - A(w)]^{-1}\bar{B}(w) \\ &= [0 \dots 0 \ 1] \begin{bmatrix} s & 0 & \dots & 0 & -p_n \\ -p_1 & s & \dots & 0 & -p_{n+1} \\ 0 & -p_2 & \dots & 0 & -p_{n+2} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & -p_{n-1} & s - p_{2n-1} \end{bmatrix}^{-1} \begin{bmatrix} \bar{b}_0 \\ \vdots \\ \bar{b}_{n-1} \end{bmatrix} \\ &= \frac{R_n(w)\bar{B}(w)}{s^n - a_{n-1}(w)s^{n-1} - \dots - a_1(w)s - a_0(w)} \\ &= \frac{n(s, w)}{d(s, w)}. \end{aligned} \quad (37)$$

**Theorem 2.** A positive realization

$$\begin{aligned} A_0 &\in M_n, \quad A_i \in \mathbb{R}_+^{n \times n}, \quad i = 1, 2, \dots, h, \\ B_j &\in \mathbb{R}_+^{n \times m}, \quad j = 0, 1, \dots, q, \\ C &= [0 \dots 0 \ 1] \in \mathbb{R}_+^{1 \times n} \end{aligned} \quad (38)$$

of the transfer function (32) exists if it is possible to find polynomials

$$p_1(w), \quad p_2(w), \dots, \quad p_{2n-1}(w) \quad (39)$$

and

$$\bar{b}_0(w), \quad \bar{b}_1(w), \dots, \quad \bar{b}_{n-1}(w) \quad (40)$$

with nonnegative coefficients (except the last coefficient of  $p_{2n-1}(w)$  such that (33) and (36) are satisfied.

*Proof.* If the polynomials (39) have nonnegative coefficients (except the last coefficient of  $p_{2n-1}(w)$ ), then

$$A(w) = P(w) = A_h w^h + \dots + A_1 w + A_0 \quad (41)$$

and  $A_0 \in M_n, A_i \in \mathbb{R}_+^{n \times n}, i = 1, 2, \dots, h$ . If the coefficients of the polynomials (40) are nonnegative, then

$$\bar{B}(w) = \begin{bmatrix} \bar{b}_0(w) \\ \vdots \\ \bar{b}_{n-1}(w) \end{bmatrix} = B_q w^q + \dots + B_1 w + B_0 \quad (42)$$

and  $B_j \in \mathbb{R}_+^{n \times m}, j = 0, 1, \dots, q$ . The matrices (38) are a realization of (32) since they satisfy (37). ■

If the assumption of Theorem 2 is met, then the positive realization (38) of (32) can be found by using the following procedure.

**Procedure 1.**

*Step 1.* Knowing the coefficients  $a_k(w), b_k(w), k = 0, 1, \dots, n-1$  of (32), find the polynomials (39) and

(40) with nonnegative coefficients satisfying the conditions (33) and (36).

**Step 2.** Knowing (39) and using (41), find the matrices  $A_k$  for  $k = 0, 1, \dots, h$ .

**Step 3.** Using the equalities (36) and (42), find the polynomials  $\bar{b}_0(w), \bar{b}_1(w), \dots, \bar{b}_{n-1}(w)$  and the matrices  $B_j$ ,  $j = 0, 1, \dots, q$ .

**Example 1.** Using Procedure 1, find a positive realization of the transfer function

$$T_{sp}(s, w) = \frac{(w^2 + 2w)s + (w^3 + w^2)}{s^2 - (2w - 3)s - (w^3 + w)}. \quad (43)$$

**Step 1.** In this case we choose the polynomials

$$p_1(w) = w, \quad p_2(w) = w^2 + 1, \quad p_3(w) = 2w - 3 \quad (44)$$

and

$$\bar{b}_0(w) = w^2 + w, \quad \bar{b}_1(w) = w^2 + 2w, \quad (45)$$

which satisfy the conditions (33) and (36) since

$$\begin{aligned} \det[I_2 s - P(w)] &= \begin{vmatrix} s & -p_2 \\ -p_1 & s - p_3 \end{vmatrix} \\ &= s^2 - p_3 s - p_1 p_2 = s^2 - (2w - 3)s - (w^3 + w) \end{aligned}$$

and

$$\begin{aligned} \begin{bmatrix} p_1 & s \end{bmatrix} \begin{bmatrix} \bar{b}_0(w) \\ \bar{b}_1(w) \end{bmatrix} &= \begin{bmatrix} w & s \end{bmatrix} \begin{bmatrix} w^2 + w \\ w^2 + 2w \end{bmatrix} \\ &= (w^2 + 2w)s + (w^3 + w^2). \end{aligned} \quad (46)$$

**Step 2.** Using (41) and (44), we obtain

$$A(w) = P(w) = \begin{bmatrix} 0 & w^2 + 1 \\ w & 2w - 3 \end{bmatrix} = A_0 + A_1 w + A_2 w^2, \quad (47)$$

where

$$\begin{aligned} A_0 &= \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \end{aligned} \quad (48)$$

**Step 3.** Taking into account that

$$\begin{aligned} \bar{B}(w) &= \begin{bmatrix} \bar{b}_0(w) \\ \bar{b}_1(w) \end{bmatrix} = \begin{bmatrix} w^2 + w \\ w^2 + 2w \end{bmatrix} \\ &= B_0 + B_1 w + B_2 w^2, \end{aligned} \quad (49a)$$

where

$$B_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad (49b)$$

the desired positive realization of (43) is given by (48), (4.26b) and

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = [0]. \quad (49c)$$

The proposed method can be extended to a Multi-Input Multi-Output (MIMO) linear system with delays.

It is well known that the proper transfer matrix of MIMO linear systems with delays can be written in the form

$$\begin{aligned} T(s, w) &= \begin{bmatrix} \frac{n_{11}(s, w)}{d_1(s, w)} & \cdots & \frac{n_{1,m}(s, w)}{d_1(s, w)} \\ \vdots & \cdots & \vdots \\ \frac{n_{p,1}(s, w)}{d_p(s, w)} & \cdots & \frac{n_{p,m}(s, w)}{d_p(s, w)} \end{bmatrix} \\ &+ D \in \mathbb{R}^{p \times m}(s, z), \end{aligned} \quad (50a)$$

where

$$\begin{aligned} n_{i,j}(s, w) &= b_{n_i-1}^{i,j}(w)s^{n_i-1} + \cdots + b_1^{i,j}(w)s + b_0^{i,j}(w), \\ i &= 1, 2, \dots, p, \quad j = 1, 2, \dots, m, \\ b_k^{i,j}(w) &= b_{k,q_i}^{i,j} w^{q_i} + \cdots + b_{k,1}^{i,j} w + b_{k,0}^{i,j}, \\ k &= 0, 1, \dots, n_i - 1. \end{aligned} \quad (50b)$$

**Theorem 3.** A positive realization

$$\begin{aligned} A_0 &= \text{blockdiag} [A_{01}, \dots, A_{0,p}] \in M_n, \\ n &= n_1 + \cdots + n_p, \end{aligned} \quad (51a)$$

$$\begin{aligned} A_k &= \text{blockdiag} [A_{k,1}, \dots, A_{k,p}] \in \mathbb{R}_+^{n \times n}, \\ k &= 1, 2, \dots, h, \end{aligned} \quad (51b)$$

$$\begin{aligned} B_k &= \begin{bmatrix} B_{11}^k & \cdots & B_{1,m}^k \\ \vdots & \cdots & \vdots \\ B_{p,1}^k & \cdots & B_{p,m}^k \end{bmatrix} \in \mathbb{R}_+^{n \times m}, \\ k &= 0, 1, \dots, q, \end{aligned} \quad (51c)$$

$$\begin{aligned} C &= \text{blockdiag} [C_1, \dots, C_p] \in \mathbb{R}_+^{p \times n}, \\ C_i &= \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \in \mathbb{R}_+^{1 \times n_i}, \quad i = 1, 2, \dots, p, \end{aligned} \quad (51d)$$

of the strictly proper transfer matrix (50) if it is possible to find the polynomials

$$p_1^i(w), p_2^i(w), \dots, p_{2n_i-1}^i(w), \quad i = 1, 2, \dots, p \quad (52a)$$

and

$$\begin{aligned} \bar{b}_0^{i,j}(w), \bar{b}_1^{i,j}(w), \dots, \bar{b}_{n_i-1}^{i,j}(w), \\ i = 1, 2, \dots, p, \quad j = 1, 2, \dots, m, \end{aligned} \quad (52b)$$

with nonnegative coefficients (except the last coefficient of  $p_{2n_i-1}^i(w)$ ) such that the conditions

$$\begin{aligned} & \det[I_{n_i}s - P_i(w)] \\ &= \begin{vmatrix} s & 0 & \dots & 0 & -p_{n_i}^i \\ -p_1^i & s & \dots & 0 & -p_{n_i+1}^i \\ 0 & -p_2^i & \dots & 0 & -p_{n_i+2}^i \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & -p_{n_i-1}^i & s - p_{2n_i-1}^i \end{vmatrix} \quad (53) \\ &= d_i(s, w), \quad i = 1, 2, \dots, p, \end{aligned}$$

$$\begin{aligned} & p_1^i p_2^i \dots p_{n_i-1}^i \bar{b}_0^{i,j} + p_2^i p_3^i \dots p_{n_i-1}^i \bar{b}_1^{i,j} s \\ &+ \dots + p_{n_i-1}^i \bar{b}_{n_i-2}^{i,j} s^{n_i-2} + \bar{b}_{n_i-1}^{i,j} s^{n_i-1} \\ &= b_0^{i,j} + b_1^{i,j} s + \dots + b_{n_i-2}^{i,j} s^{n_i-2} + b_{n_i-1}^{i,j} s^{n_i-1} \quad (54) \\ &= n_{i,j}(s, w), \\ & \quad i = 1, 2, \dots, p, \quad j = 1, 2, \dots, m, \end{aligned}$$

are satisfied.

*Proof.* If the polynomials (52a) have nonnegative coefficients (except the last coefficient of  $p_{2n_i-1}^i$ ), then

$$A_k(w) = P_k(w) = \text{blockdiag} [A_{k,1}, \dots, A_{k,p}], \quad k = 0, 1, \dots, h, \quad (55)$$

and (51a) and (51b) hold. If the coefficients of the polynomials (52b) are nonnegative, then

$$\begin{aligned} B_{i,j}(w) &= B_{i,j}^0 + B_{i,j}^1 w + \dots + B_{i,j}^q w^q, \\ & \quad i = 1, 2, \dots, p, \quad j = 1, 2, \dots, m, \end{aligned} \quad (56)$$

and (51c) is satisfied. It is easy to check that the matrices (51) and  $D \in \mathbb{R}_+^{p \times m}$  satisfy the equality

$$C[I_n s - A]^{-1} B + D = T(s, w), \quad (57)$$

where  $T(s, w)$  is given by (50a). ■

If the conditions of Theorem 3 are satisfied, then the positive realization of (50a) can be found by using the following procedure.

#### Procedure 2.

*Step 1.* Using the formula

$$D = \lim_{s \rightarrow \infty} T(s, w),$$

find the matrix  $D$  and the strictly proper part  $T_{sp}(s, w)$  of (50a).

*Step 2.* Choose the polynomials (52) with nonnegative coefficients (except the last coefficient of  $p_{2n_i-1}^i$ ) satisfying (53) and (54).

*Step 3.* Using (55), find the matrices (51a) and (51b).

*Step 4.* Using (56), find the matrices (51c) and (51d).

**Example 2.** Find a positive realization of the strictly proper transfer matrix

$$T_{sp}(s, w) = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \quad (58a)$$

where  $T_{11}, T_{12}, T_{21}$  and  $T_{22}$  are given by (58b).

Using Procedure 2, we obtain the following:

*Step 1.* The transfer matrix (58) already has the desired form (50) with  $D = 0$ .

*Step 2.* For the first row of (58), we choose

$$\begin{aligned} p_1^1 &= w^2, & p_2^1 &= w + 1, & p_3^1 &= w + 2, \\ p_4^1 &= w, & p_5^1 &= w^2 - 3, \end{aligned} \quad (59a)$$

and for the second row,

$$p_1^2 = w^2 + 1, \quad p_2^2 = w + 1, \quad p_3^2 = w^2 - 2, \quad (59b)$$

since

$$\begin{aligned} \det[I_3 s - P_1(w)] &= \begin{vmatrix} s & 0 & -(w+2) \\ -w^2 & s & -w \\ 0 & -(w+1) & s - w^2 + 3 \end{vmatrix} \\ &= s^3 - (w^2 - 3)s^2 - (w^2 + w)s - (w^4 + 3w^3 + 2w^2), \end{aligned} \quad (60a)$$

$$\begin{aligned} \det[I_2 s - P_2(w)] &= \begin{vmatrix} s & -(w+1) \\ -(w^2 + 1) & s - w^2 + 2 \end{vmatrix} \\ &= s^2 - (w^2 - 2)s - (w^3 + w^2 + w + 1) \end{aligned} \quad (60b)$$

and

$$\begin{aligned} & [p_1^1 p_2^1 \quad p_1^2 p_2^2 \quad s^2] \begin{bmatrix} \bar{b}_0^{1,1}(w) \\ \bar{b}_1^{1,1}(w) \\ \bar{b}_2^{1,1}(w) \end{bmatrix} \\ &= [w^3 + w^2 \quad (w+1)s \quad s^2] \begin{bmatrix} 1 \\ w \\ w^2 + 2 \end{bmatrix} \\ &= (w^2 + 2)s^2 + (w^2 + w)s + w^3 + w^2, \end{aligned} \quad (61a)$$

$$\begin{aligned} & [p_1^1 p_2^1 \quad p_1^2 p_2^2 \quad s^2] \begin{bmatrix} \bar{b}_0^{1,2}(w) \\ \bar{b}_1^{1,2}(w) \\ \bar{b}_2^{1,2}(w) \end{bmatrix} \\ &= [w^3 + w^2 \quad (w+1)s \quad s^2] \begin{bmatrix} 2w \\ w^2 \\ w^3 + w \end{bmatrix} \\ &= (w^3 + w)s^2 + (w^3 + w^2)s + 2w^4 + 2w^3, \end{aligned} \quad (61b)$$

$$\begin{aligned} & [p_1^2 \quad s] \begin{bmatrix} \bar{b}_0^{2,1}(w) \\ \bar{b}_1^{2,1}(w) \end{bmatrix} = [w^2 + 1 \quad s] \begin{bmatrix} w \\ w^2 + 2 \end{bmatrix} \\ &= (w^2 + 2)s + w^3 + w, \end{aligned} \quad (62a)$$

$$\begin{aligned} & [p_1^2 \quad s] \begin{bmatrix} \bar{b}_0^{2,2}(w) \\ \bar{b}_1^{2,2}(w) \end{bmatrix} = [w^2 + 1 \quad s] \begin{bmatrix} w + 1 \\ w^3 \end{bmatrix} \\ &= w^3 s + w^3 + w^2 + w + 1. \end{aligned} \quad (62b)$$



$$\begin{aligned}
 T_{11} &= \frac{(w^2 + 2)s^2 + (w^2 + w)s + w^3 + w^2}{s^3 - (w^2 - 3)s^2 - (w^2 + w)s - (w^4 + 3w^3 + 2w^2)}, \\
 T_{12} &= \frac{(w^3 + w)s^2 + (w^3 + w^2)s + 2w^4 + 2w^3}{s^3 - (w^2 - 3)s^2 - (w^2 + w)s - (w^4 + 3w^3 + 2w^2)}, \\
 T_{21} &= \frac{(w^2 + 2)s + w^3 + w}{s^2 - (w^2 - 2)s - (w^3 + w^2 + w + 1)}, \\
 T_{22} &= \frac{w^3s + w^3 + w^2 + w + 1}{s^2 - (w^2 - 2)s - (w^3 + w^2 + w + 1)}. \quad (58b)
 \end{aligned}$$

Step 3. Using (55) and (59), we obtain

$$\begin{aligned}
 A_1(w) &= P_1(w) = A_{10} + A_{11}w + A_{12}w^2, \\
 A_{10} &= \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & -3 \end{bmatrix}, \quad A_{11} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \\
 A_{12} &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (63a)
 \end{aligned}$$

$$\begin{aligned}
 A_2(w) &= P_2(w) = A_{20} + A_{21}w + A_{22}w^2, \\
 A_{20} &= \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\
 A_{22} &= \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \quad (63b)
 \end{aligned}$$

and

$$A_k = \text{blockdiag} [A_{k,1}, A_{k,2}], \quad k = 0, 1, 2. \quad (64)$$

Step 4. Using (56), (61) and (62), we obtain

$$\begin{aligned}
 \bar{B}_{11}(w) &= \begin{bmatrix} 1 \\ w \\ w^2 + 2 \end{bmatrix} = B_{11}^0 + B_{11}^1w + B_{11}^2w^2, \\
 B_{11}^0 &= \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad B_{11}^1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \\
 B_{11}^2 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad B_{11}^3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (65a)
 \end{aligned}$$

$$\begin{aligned}
 \bar{B}_{12}(w) &= \begin{bmatrix} 2w \\ w^2 \\ w^3 + w \end{bmatrix} \\
 &= B_{12}^0 + B_{12}^1w + B_{12}^2w^2 + B_{12}^3w^3, \\
 B_{12}^0 &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_{12}^1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \\
 B_{12}^2 &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad B_{12}^3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad (65b)
 \end{aligned}$$

$$\begin{aligned}
 \bar{B}_{21}(w) &= \begin{bmatrix} w \\ w^2 + 2 \end{bmatrix} = B_{21}^0 + B_{21}^1w + B_{21}^2w^2, \\
 B_{21}^0 &= \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad B_{21}^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
 B_{21}^2 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_{21}^3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (65c)
 \end{aligned}$$

$$\begin{aligned}
 \bar{B}_{22}(w) &= \begin{bmatrix} w + 1 \\ w^3 \end{bmatrix} \\
 &= B_{22}^0 + B_{22}^1w + B_{22}^2w^2 + B_{22}^3w^3, \\
 B_{22}^0 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_{22}^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
 B_{22}^2 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad B_{22}^3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (65d)
 \end{aligned}$$

$$B_k = \begin{bmatrix} B_{11}^k & B_{12}^k \\ B_{21}^k & B_{22}^k \end{bmatrix}, \quad k = 0, 1, 2, 3. \quad (66)$$

The desired positive realization of (58) is given by (64), (66) and

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad (67)$$

## 5. Concluding remarks

A new modified state variable diagram method for determination of positive realizations of linear continuous-time



systems with delays in state and input vectors has been proposed. Using the method it is possible to find a positive realization with reduced numbers of delays for given proper transfer matrices. Sufficient conditions for the existence of positive realizations have been established and procedures for finding the positive realizations have been proposed. The procedures have been illustrated by numerical examples. The proposed method can be extended to continuous-discrete linear systems and to fractional continuous-time linear ones.

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