VERSITA

Int. J. Appl. Math. Comput. Sci., 2012, Vol. 22, No. 3, 561–574
DOI: 10.2478/v10006-012-0043-4

amcs

ENCLOSURES FOR THE SOLUTION SET OF PARAMETRIC INTERVAL LINEAR SYSTEMS

MILAN HLADÍK

Department of Applied Mathematics, Faculty of Mathematics and Physics
Charles University, Malostranské nám. 25, 118 00, Prague, Czech Republic
e-mail: milan.hladik@matfyz.cz

Faculty of Informatics and Statistics
University of Economics, nám. W. Churchilla 4, 13067, Prague, Czech Republic

We investigate parametric interval linear systems of equations. The main result is a generalization of the Bauer–Skeel and the Hansen–Bliek–Rohn bounds for this case, comparing and refinement of both. We show that the latter bounds are not provable better, and that they are also sometimes too pessimistic. The presented form of both methods is suitable for combining them into one to get a more efficient algorithm. Some numerical experiments are carried out to illustrate performances of the methods.

Keywords: linear interval systems, solution set, interval matrix.

1. Introduction

Solving systems of interval linear equations is a fundamental problem in interval computing (Fiedler *et al.*, 2006; Neumaier, 1990). Therein, one assumes that the matrix entries and the right-hand side components perturb independently and simultaneously within given intervals. However, this assumption is hardly true in practical problems. Very often various correlations between input quantities appear, e.g., in robotics (Merlet, 2009) or in dynamic systems (Busłowicz, 2010).

Linear dependences were investigated by several authors. The first paper on parametric interval systems (with a special structure) is that by Jansson (1991). For a special class of parametric systems, Neumaier and Pownuk (2007) proposed an effective method. The general problem of interval parameter dependent linear systems was first treated by Rump (1994).

Theoretical papers involve, e.g., characterization of the boundary of the solution set (Popova and Krämer, 2008), the quality of the solution set (Popova, 2002), or an explicit characterization of a class of parametric interval systems (Hladík, 2008; Popova, 2009). Shapes of the particular solution sets were first analyzed by Alefeld *et al.* (1997; 2003).

Kolev (2006) proposed a direct method and an iterative one (Kolev, 2004) for computing an enclosure of the solution set. Parametrized Gauss–Seidel iteration was employed by Popova (2001). A direct method was given by Skalna (2006), and a monotonicity approach by Popova (2006a), Rohn (2004), and Skalna (2008). Inner and outer approximations by a fixed-point method were developed by Rump (1994; 2010), and implemented by Popova and Krämer (2007). A *Mathematica* package for solving parametric interval systems is introduced by Popova (2004a).

Let

$$\boldsymbol{p} := [\underline{p}, \overline{p}] = \{p \in \mathbb{R}^K \mid \underline{p} \le p \le \overline{p}\}$$

be an interval vector. By $p^c := \frac{1}{2}(\overline{p} + \underline{p})$ and $p^\Delta := \frac{1}{2}(\overline{p} - \underline{p})$ we denote the corresponding center and the radius vector. Analogous notation is used for interval matrices. We suppose that the reader is familiar with the basic interval arithmetic.

In this paper, we consider a general parametric system of interval linear equations in the form

$$A(p)x = b(p), \quad p \in \boldsymbol{p}, \tag{1}$$

where

$$A(p) = \sum_{k=1}^{K} p_k A^k, \quad b(p) = \sum_{k=1}^{K} p_k b^k.$$

Herein, $\boldsymbol{p}$ is the interval vector representing K interval parameters, and $A^k \in \mathbb{R}^{n\times n}$ and $b^k \in \mathbb{R}^n$, $k = 1,\dots,K$, are given matrices and vectors. Notice that this linear parametric form comprises affine linear parametric matrices and vectors,

$$A^0 + \sum_{k=1}^{K} p_k A^k, \quad b^0 + \sum_{k=1}^{K} p_k b^k,$$

since one can simply introduce an idle parameter $p_0 := [1,1]$. In our approach, no better results are obtained explicitly for the affine linear parametric structure.

The solution set is defined as

$$\Sigma := \{x \in \mathbb{R}^n \mid A(p)x = b(p),\ p \in \boldsymbol{p}\}.$$

We use the following notation: $\rho(A)$ stands for the spectral radius of a matrix A, $A_{i\bullet}$ for the i-th row of A, I for the identity matrix and e_i for its ith column. The diagonal matrix with entries $z_1,\dots,z_n$ is denoted by $\mathrm{diag}(z)$, and $A(\boldsymbol{p})$ is a short form for a family $A(p)$, $p \in \boldsymbol{p}$. We write interval quantities in boldface.

The paper is structured as follows. In Section 2 we discuss the regularity of a parametric interval matrix, and in Section 3 enclosures of a parametric interval linear system. We generalize the Bauer–Skeel and the Hansen–Bliek–Rohn bounds, which were developed for a standard interval linear system; for the reader's convenience, we recall the original formulae in Appendix. Moreover, we propose efficient refinements of both methods.

2. Regularity of parametric interval matrices

In order to develop an enclosure for the parametric interval system we have to discuss the regularity of the parametric interval matrix $A(\boldsymbol{p})$ first. The parametric interval matrix is called *regular* if $A(p)$ is nonsingular for every $p \in \boldsymbol{p}$.

Preconditioning and relaxing the parametric interval matrix, we obtain an interval matrix

$$\boldsymbol{A} = \sum_{k=1}^{K} \boldsymbol{p}_k (RA^k),$$

i.e.,

$$\boldsymbol{A}_{ij} = \Bigg[\sum_{k=1}^{K} \min\left(\underline{p}_k (RA^k)_{ij}, \overline{p}_k (RA^k)_{ij}\right), \sum_{k=1}^{K} \max\left(\underline{p}_k (RA^k)_{ij}, \overline{p}_k (RA^k)_{ij}\right)\Bigg].$$

Clearly, if $\boldsymbol{A}$ is regular, then so is $A(\boldsymbol{p})$. Thus we can employ the well-known Beeck–Rump sufficient condition for the regularity of interval matrices (Beeck, 1975; Rump, 1983; Rex and Rohn, 1998).

Theorem 1. *Let $R \in \mathbb{R}^{n\times n}$ be such that*

$$\rho\left(|I - RA(p^c)| + \sum_{k=1}^{K} p_k^{\Delta} |RA^k|\right) < 1. \tag{2}$$

Then $A(\boldsymbol{p})$ is regular.

Usually, the best choice for the matrix R is the numerically computed inverse of $A(p^c)$. In the following, we consider the case $R = A(p^c)^{-1}$. For this special case, the sufficient condition was already stated by Popova (2004b).

How strong is the sufficient condition presented in Theorem 1? The following result shows a class of problems where the condition is not only sufficient, but also necessary. It is a generalization of Rohn's result (Rohn, 1989, Corollary 5.1.(ii)).

Proposition 1. *Suppose that $A(p^c)$ is nonsingular and there are $z \in \{\pm 1\}^n$ and $y \in \{\pm 1\}^K$ such that for every $k \in \{1,\dots,K\}$ we have*

$$y_k \,\mathrm{diag}(z)\, A(p^c)^{-1} A^k \,\mathrm{diag}(z) \geq 0.$$

Then $A(\boldsymbol{p})$ is regular if and only if

$$\rho\left(\sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1} A^k|\right) < 1.$$

Proof. One implication is obvious in view of Theorem 1. We prove the converse by contradiction. Denote

$$\begin{aligned} A^* &:= \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1} A^k| \\ &= \sum_{k=1}^{K} p_k^{\Delta} y_k \,\mathrm{diag}(z)\, A(p^c)^{-1} A^k \,\mathrm{diag}(z), \end{aligned}$$

and suppose for contradiction that $\rho(A^*) \geq 1$. Since A^* is non-negative, according to the Perron–Frobenius theorem (Horn and Johnson, 1985; Meyer, 2000) there is some non-zero vector x such that

$$A^* x = \rho(A^*) x$$

or, equivalently,

$$\left(I - \frac{1}{\rho(A^*)} A^*\right) x = 0.$$

Premultiplying by $A(p^c)\,\mathrm{diag}(z)$, we get

$$\left(A(p^c)\,\mathrm{diag}(z) - \frac{1}{\rho(A^*)} A(p^c)\,\mathrm{diag}(z)\, A^*\right) x = 0$$

or

$$\left(\sum_{k=1}^{K} \left(p_k^c - \frac{y_k}{\rho(A^*)} p_k^{\Delta}\right) A^k\right) (\mathrm{diag}(z)\, x) = 0.$$

The vector $\operatorname{diag}(z)\, x$ is non-zero, and the constraint matrix belongs to $A(\boldsymbol{p})$ since

$$p_k^c - \frac{y_k}{\rho(A^*)} p_k^\Delta \in \boldsymbol{p}_k, \quad k = 1, \dots, K.$$

Thus we found a singular matrix in $A(\boldsymbol{p})$, which is a contradiction. ■

3. Enclosures for parametric interval linear systems

The main problem studied within this paper is to find a tight enclosure for the solution set Σ, where an enclosure is any interval vector containing Σ. A simple enclosure can be acquired by relaxing the system (1) to an interval linear system $\boldsymbol{A}x = \boldsymbol{b}$, where (by using interval arithmetic)

$$\boldsymbol{A} := \sum_{k=1}^{K} \boldsymbol{p}_k A^k, \quad \boldsymbol{b} := \sum_{k=1}^{K} \boldsymbol{p}_k b^k.$$

Since many efficient solvers of interval linear systems use preconditioning, we should note that instead of preconditioning the system $\boldsymbol{A}x = \boldsymbol{b}$ by a matrix R it is better to precondition the original data. That is, consider $\boldsymbol{A}'x = \boldsymbol{b}'$, where

$$\boldsymbol{A}' := \sum_{k=1}^{K} \boldsymbol{p}_k (RA^k), \quad \boldsymbol{b}' := \sum_{k=1}^{K} \boldsymbol{p}_k (Rb^k). \tag{3}$$

Proposition 2. *We have $\boldsymbol{A}' \subseteq R\boldsymbol{A}$ and $\boldsymbol{b}' \subseteq R\boldsymbol{b}$.*

Proof. Let $i, j \in \{1, \dots, n\}$. Due to the sub-distributivity of interval arithmetic, we can write

$$\begin{aligned}
\boldsymbol{A}'_{ij} &= \sum_{k=1}^{K} \boldsymbol{p}_k (RA^k)_{ij} = \sum_{k=1}^{K} \boldsymbol{p}_k \left(\sum_{l=1}^{n} R_{il} A_{lj}^k \right) \\
&\subseteq \sum_{k=1}^{K} \sum_{l=1}^{n} \boldsymbol{p}_k R_{il} A_{lj}^k = \sum_{l=1}^{n} \sum_{k=1}^{K} R_{il} (\boldsymbol{p}_k A_{lj}^k) \\
&= \sum_{l=1}^{n} R_{il} \left(\sum_{k=1}^{K} \boldsymbol{p}_k A_{lj}^k \right) = \sum_{l=1}^{n} R_{il} \boldsymbol{A}_{lj} = (R\boldsymbol{A})_{ij}.
\end{aligned}$$

We proceed similarly for $\boldsymbol{b}' \subseteq R\boldsymbol{b}$. ■

To obtain tighter enclosures, we have to inspect parametric systems more carefully. Recently, Popova (2009) proved that the inequality system given below in (4) is an explicit description of a parametric interval linear system of the so-called zero or first class; in this class, for each $k = 1, \dots, K$, the nonzero entries of $(A^k \mid b^k)$ are situated in one row only. First we show this is a necessary (but not sufficient in general) characterization for any parametric interval linear system.

Theorem 2. *If $x \in \mathbb{R}^n$ solves (1) for some $p \in \boldsymbol{p}$, then it solves*

$$|A(p^c)x - b(p^c)| \le \sum_{k=1}^{K} p_k^\Delta |A^k x - b^k|. \tag{4}$$

Proof. Let $x \in \mathbb{R}^n$ be a solution to $A(p)x = b(p)$ for some $p \in \boldsymbol{p}$. Then, in a similar way as for the well known Oettli–Prager theorem, we derive

$$\begin{aligned}
&|A(p^c)x - b(p^c)| \\
&\quad = \left| \sum_{k=1}^{K} p_k^c (A^k x - b^k) \right| \\
&\quad = \left| \sum_{k=1}^{K} p_k^c (A^k x - b^k) - \sum_{k=1}^{K} p_k (A^k x - b^k) \right| \\
&\quad = \left| \sum_{k=1}^{K} (p_k^c - p_k)(A^k x - b^k) \right| \\
&\quad \le \sum_{k=1}^{K} |p_k^c - p_k| |A^k x - b^k| \\
&\quad \le \sum_{k=1}^{K} p_k^\Delta |A^k x - b^k|.
\end{aligned}$$

■

A sufficient and necessary characterization of Σ is given below in terms of infinite systems of inequalities. From another viewpoint, the system is composed of a union of systems (4) over all possible preconditionings of (1). An open question arises whether or not particular extremal points of Σ can be achieved by an appropriate preconditioning of (1).

Theorem 3. *We have that $x \in \Sigma$ if and only if it solves*

$$y^T \left(A(p^c)x - b(p^c) \right) \le \sum_{k=1}^{K} p_k^\Delta |y^T (A^k x - b^k)| \tag{5}$$

for every $y \in \mathbb{R}^n$.

Proof. Let $x \in \mathbb{R}^n$. Then $x \in \Sigma$ if and only if there is a vector $q \in [-1, 1]^K$ such that

$$A(p^c)x - b(p^c) = \sum_{k=1}^{K} q_k p_k^\Delta (A^k x - b^k).$$

Set $d := A(p^c)x - b(p^c)$, and let $D \in \mathbb{R}^{n \times K}$ be a matrix whose k-th column is equal to $p_k^\Delta (A^k x - b^k)$, $k = 1, \dots, K$. Then $x \in \Sigma$ if and only if there is an optimal solution to the linear system

$$Dq = d, \quad -1 \le q \le 1,$$

or, in other words, if and only if the linear program

$$\max\ 0^T q \ \text{ subject to } \ Dq = d,\ -1 \le q \le 1$$

has an optimal solution. Consider the corresponding dual problem

$$\min\ d^T y + 1^T(u+v)$$

subject to

$$D^T y + u - v = 0,\ u, v \ge 0,$$

which is always feasible. According to the theory of duality in linear programming (Padberg, 1999; Schrijver, 1998), the existence of an optimal solution to one problem implies the same for the second one and the optimal values are equal.

For an optimal solution of the dual problem and every $i \in \{1, \dots, K\}$ either $u_i = 0$ or $v_i = 0$. Otherwise, we can subtract a small positive amount from both u_i and v_i and decrease the optimal value. If $u_i = 0$, then $(u+v)_i = v_i = (D^T y)_i \ge 0$. Similarly, $v_i = 0$ implies $(u+v)_i = u_i = -(D^T y)_i \ge 0$. Hence we can derive $u+v = |D^T y|$, and the dual problem takes the form

$$\min\ d^T y + 1^T |D^T y| \ \text{ subject to } \ y \in \mathbb{R}^n.$$

Since the objective function is positive homogeneous, the problem has an optimal solution (equal to zero) if and only if the objective function is non-negative, i.e.,

$$d^T y + 1^T |D^T y| \ge 0, \quad \forall y \in \mathbb{R}^n$$

or, substituting $y := -y$,

$$y^T d \le |y^T D| 1, \quad \forall y \in \mathbb{R}^n.$$

In the setting of D and d, we get (5). ∎

Based on Theorem 2 we develop a generalization of the Bauer–Skeel bounds (Rohn, 2010; Stewart, 1998) to parametric interval systems. Note that the generalized Bauer–Skeel bounds yield the same enclosure as the direct method by Skalna (2006). However, the following form is more convenient for combining it with the Hansen–Bliek–Rohn method and for refinements.

Theorem 4. *Suppose that $A(p^c)$ is nonsingular. Write*

$$M := \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1} A^k|,$$
$$x^* := A(p^c)^{-1} b(p^c).$$

If $\rho(M) < 1$, then

$$\left[x^* - (I-M)^{-1} \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}(A^k x^* - b^k)|, \right.$$
$$\left. x^* + (I-M)^{-1} \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}(A^k x^* - b^k)| \right]$$

is an interval enclosure to Σ.

Proof. Preconditioning the system $A(p)x = b(p)$ by the matrix $A(p^c)^{-1}$, we obtain an equivalent system $A(p^c)^{-1}A(p)x = A(p^c)^{-1}b(p)$, or

$$\sum_{k=1}^{K} p_k A(p^c)^{-1} A^k x = \sum_{k=1}^{K} p_k A(p^c)^{-1} b^k, \quad p \in \boldsymbol{p}.$$

According to Theorem 2 each solution to this system satisfies

$$|A(p^c)^{-1}A(p^c)x - A(p^c)^{-1}b(p^c)|$$
$$\le \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}(A^k x - b^k)|.$$

Rearranging the system, we get

$$\begin{aligned} |x - x^*| &\le \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}(A^k x - b^k)| \qquad (6)\\ &= \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}(A^k (x - x^* + x^*) - b^k)| \\ &\le \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1} A^k (x - x^*)| \\ &\quad + \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}(A^k x^* - b^k)| \\ &\le \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1} A^k| |x - x^*| \\ &\quad + \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}(A^k x^* - b^k)|. \end{aligned}$$

Equivalently,

$$\left(I - \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1} A^k| \right) |x - x^*|$$
$$\le \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}(A^k x^* - b^k)|.$$

From $\rho(M) < 1$, it follows (Fiedler *et al.*, 2006; Meyer, 2000, Theorem 1.31) that

$$(I - M)^{-1} = \sum_{j=0}^{\infty} M^j.$$

Since the matrix M is non-negative, so is $(I - M)^{-1}$. Thus we may multiply the system by $(I - M)^{-1}$ to obtain

$$|x - x^*| \le (I - M)^{-1} \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}(A^k x^* - b^k)|.$$

This means, that

$$x \geq x^* - (I - M)^{-1} \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}(A^k x^* - b^k)|,$$
$$x \leq x^* + (I - M)^{-1} \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}(A^k x^* - b^k)|.$$

■

The Hansen–Bliek–Rohn method (Fiedler *et al.*, 2006; Rohn, 1993, Theorem 2.39) gives an enclosure for the solution set of an interval linear system. The following is a generalization to parametric interval linear systems; however, the result is the same as the Hansen–Bliek–Rohn bounds applied on the preconditioned system (3) by $R := A(p^c)^{-1}$. For the reader's convenience, we present a detailed proof, which will be followed up in the next section for a refinement. Note that an alternative form of the enclosure was developed by Neumaier (1999) as well as Ning and Kearfott (1997).

Theorem 5. *Suppose that $A(p^c)$ is nonsingular. Using the notation from Theorem 4, write*

$$M^* := (I - M)^{-1},$$
$$x^0 := M^*|x^*| + \sum_{k=1}^{K} p_k^{\Delta} M^* |A(p^c)^{-1} b^k|.$$

If $\rho(M) < 1$, then any solution x to (1) satisfies

$$x_i \leq \max\left\{ x_i^0 + (x_i^* - |x_i^*|)m_{ii}^*, \; \frac{1}{2m_{ii}^* - 1}\big(x_i^0 + (x_i^* - |x_i^*|)m_{ii}^*\big) \right\},$$

and

$$x_i \geq \min\left\{ -x_i^0 + (x_i^* + |x_i^*|)m_{ii}^*, \; \frac{1}{2m_{ii}^* - 1}\big(-x_i^0 + (x_i^* + |x_i^*|)m_{ii}^*\big) \right\}.$$

Proof. From the proof of Theorem 4 we know that each solution to (1) satisfies

$$|x - x^*| \leq \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}(A^k x - b^k)| \tag{7}$$
$$\leq \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1} A^k||x| + \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1} b^k|.$$

This inequality system implies

$$x - x^* \leq \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1} A^k||x| + \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1} b^k|, \tag{8}$$

and

$$|x| - |x^*| \leq \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1} A^k||x| + \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1} b^k|. \tag{9}$$

Let $i \in \{1, \dots, n\}$. Consider the system (9) in which the i-th inequality is replaced by the i-th inequality from (8),

$$|x| - |x^*| + (x_i - x_i^* - |x|_i + |x_i^*|)e_i \leq \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1} A^k||x| + \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1} b^k|.$$

This can be rewritten as

$$\left(I - \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1} A^k|\right)|x| + (x_i - |x|_i)e_i \leq |x^*| + (x_i^* - |x_i^*|)e_i + \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1} b^k|.$$

From $\rho(M) < 1$, it follows (Fiedler *et al.*, 2006; Meyer, 2000, Theorem 1.31) that

$$(I - M)^{-1} = \sum_{j=0}^{\infty} M^j.$$

Since the matrix M is non-negative, $M^* = (I - M)^{-1} \geq I$. Thus we may multiply the system by $M^* \geq 0$ to obtain

$$|x| + (x_i - |x|_i)M^* e_i \leq M^*|x^*| + (x_i^* - |x_i^*|)M^* e_i + \sum_{k=1}^{K} p_k^{\Delta} M^* |A(p^c)^{-1} b^k|.$$

Setting

$$x^0 = M^*|x^*| + \sum_{k=1}^{K} p_k^{\Delta} M^* |A(p^c)^{-1} b^k|,$$

the system reads

$$|x| + (x_i - |x|_i)M^* e_i \leq x^0 + (x_i^* - |x_i^*|)M^* e_i.$$

The i-th inequality becomes

$$|x_i| + (x_i - |x|_i)m_{ii}^* \leq x_i^0 + (x_i^* - |x_i^*|)m_{ii}^*.$$

Distinguish two cases. If $x_i \geq 0$, then

$$x_i \leq x_i^0 + (x_i^* - |x_i^*|)m_{ii}^*.$$

If $x_i < 0$, then

$$-x_i + 2x_i m^*_{ii} \le x^0_i + (x^*_i - |x^*_i|)m^*_{ii}.$$

Using the fact that $M^* \ge I$, we get that $2m^*_{ii} \ge 2 > 1$ and

$$x_i \le \frac{1}{2m^*_{ii} - 1}\big(x^0_i + (x^*_i - |x^*_i|)m^*_{ii}\big).$$

Summing up, we have an upper bound on x_i as follows:

$$x_i \le \max\Big\{x^0_i + (x^*_i - |x^*_i|)m^*_{ii}, \\ \frac{1}{2m^*_{ii} - 1}\big(x^0_i + (x^*_i - |x^*_i|)m^*_{ii}\big)\Big\}.$$

To obtain a lower bound on x_i, we realize that $Ax = b$ if and only if $A(-x) = -b$. Thus, we apply the previous result to the parametric interval system

$$A(p)(-x) = -b(p).$$

That is, the sign of b^c and x^* will be changed and

$$-x_i \le \max\Big\{x^0_i + (-x^*_i - |x^*_i|)m^*_{ii}, \\ \frac{1}{2m^*_{ii} - 1}\big(x^0_i + (-x^*_i - |x^*_i|)m^*_{ii}\big)\Big\},$$

or,

$$x_i \ge \min\Big\{ - x^0_i + (x^*_i + |x^*_i|)m^*_{ii}, \\ \frac{1}{2m^*_{ii} - 1}\big(- x^0_i + (x^*_i + |x^*_i|)m^*_{ii}\big)\Big\}.$$

∎

Remark 1. The Bauer–Skeel and Hansen–Bliek–Rohn methods are similar to each other since they are derived from the same basis. Nevertheless, as we will see in Section 6, both methods are incomparable, that is, sometimes the former is better and sometimes the latter. Thus, to obtain enclosure as tight as possible we propose to compute both and take their intersection. The overall computational cost is low since we calculate the inverses $A(p^c)^{-1}$, $M^* = (I - M)^{-1}$ and other intermediate expressions only once. Using notations of Theorems 4 and 5, we compute the upper endpoints of the resulting enclosure as the minima of

$$x^*_i + M^*_{i\bullet}\sum_{k=1}^{K} p^\Delta_k |A(p^c)^{-1}(A^k x^* - b^k)|,$$

and

$$\max\Big\{x^0_i + (x^*_i - |x^*_i|)m^*_{ii}, \\ \frac{1}{2m^*_{ii} - 1}\big(x^0_i + (x^*_i - |x^*_i|)m^*_{ii}\big)\Big\},$$

$i = 1, \dots, n$. We proceed similarly for the lower endpoints.

4. Refinement of enclosures

Now we show that the enclosures discussed in the previous section can be made tighter. The idea is to use those enclosures to check some sign invariances, and if they hold true, then the process of deriving the enclosures can be refined. Note that the proposed refinements run always in polynomial time.

Let $\boldsymbol{x}$ be the enclosure obtained by Theorems 4 or 5 or by any other method, and let $k \in \{1, \dots, K\}$. Write $\boldsymbol{a}^k := A(p^c)^{-1}(A^k \boldsymbol{x} - b^k)$. We will employ notations from Theorems 4 and 5, too. For the refinements, we assume $\rho(M) < 1$.

4.1. Refinement of the Bauer–Skeel bounds. First, we consider the Bauer–Skeel bounds. If $\underline{a}^k \ge 0$, then for every $x \in \Sigma$ one has

$$|A(p^c)^{-1}(A^k x - b^k)| \\ = A(p^c)^{-1}A^k(x - x^*) + A(p^c)^{-1}(A^k x^* - b^k). \quad (10)$$

Otherwise, if $\overline{a}^k \le 0$, then

$$|A(p^c)^{-1}(A^k x - b^k)| \\ = -A(p^c)^{-1}A^k(x - x^*) - A(p^c)^{-1}(A^k x^* - b^k). \quad (11)$$

Otherwise, we estimate the term from above as in the proof

$$|A(p^c)^{-1}(A^k x - b^k)| \\ \le |A(p^c)^{-1}A^k||x - x^*| + |A(p^c)^{-1}(A^k x^* - b^k)|. \quad (12)$$

Anyway, the inequality (6) can be written as

$$|x - x^*| \le \sum_{k=1}^{K} p^\Delta_k |A(p^c)^{-1}(A^k x - b^k)| \\ \le Y(x - x^*) + y + Z|x - x^*| + z$$

for some $Y, Z \in \mathbb{R}^{n\times n}$, $y, z \in \mathbb{R}^n$, and $Z \ge 0$. Here, Y and y are summed up from (10) and (11), whereas Z and z come from (12). Now, we proceed as follows:

$$|x - x^*| \le |Y||x - x^*| + y + Z|x - x^*| + z,$$

whence

$$(I - |Y| - Z)|x - x^*| \le y + z,$$

and

$$x \le x^* + (I - |Y| - Z)^{-1}(y + z), \\ x \ge x^* - (I - |Y| - Z)^{-1}(y + z).$$

Algorithm 1 (Refinement of the Bauer–Skeel method)

$Y := 0$; $y := 0$; $Z := 0$; $z := 0$;
$x^* := A(p^c)^{-1} b(p^c)$;
Let $\boldsymbol{x}$ be an initial enclosure to Σ;
for $k = 1, \dots, K$ **do**
  $\boldsymbol{a}^k := A(p^c)^{-1}(A^k \boldsymbol{x} - b^k)$;
  for $j = 1, \dots, n$ **do**
    if $\underline{a}_j^k \geq 0$ **then**
      $Y_{j\bullet} := Y_{j\bullet} + p_k^\Delta A(p^c)^{-1}_{j\bullet} A^k;\ y_j := y_j + p_k^\Delta A(p^c)^{-1}_{j\bullet}(A^k x^* - b^k)$;
    else if $\overline{a}_j^k \leq 0$ **then**
      $Y_{j\bullet} := Y_{j\bullet} - p_k^\Delta A(p^c)^{-1}_{j\bullet} A^k;\ y_j := y_j - p_k^\Delta A(p^c)^{-1}_{j\bullet}(A^k x^* - b^k)$;
    else
      $Z_{j\bullet} := Z_{j\bullet} + p_k^\Delta |A(p^c)^{-1}_{j\bullet} A^k|;\ z_j := z_j + p_k^\Delta |A(p^c)^{-1}_{j\bullet}(A^k x^* - b^k)|$;
    end if
  end for
end for
return $\left[x^* - (I - |Y| - Z)^{-1}(y + z),\ x^* + (I - |Y| - Z)^{-1}(y + z)\right]$, an enclosure to Σ.

Since $|Y| + Z$ is non-negative and $|Y| + Z \leq M$, the inverse matrix $(I - |Y| - Z)^{-1}$ exists and is non-negative.

Notice that even tighter bounds can be calculated by splitting the terms of (6) componentwise. That is, we check the signs of $\underline{a}_i^k$ and $\overline{a}_i^k$ for every $i = 1, \dots, n$, and use the i-th estimate either in (10), (11) or (12) accordingly. The method is described in Algorithm 1.

In the following we claim that the resulting enclosure is always as good as the initial Bauer–Skeel bounds.

Proposition 3. *Let $\boldsymbol{x}$ be the enclosure obtained by Theorem 4, and $\boldsymbol{x}'$ the enclosure obtained by Algorithm 1. Then $\boldsymbol{x}' \subseteq \boldsymbol{x}$.*

Proof. Recall that

$$\begin{aligned}\overline{x} &= x^* + (I - M)^{-1} \sum_{k=1}^{K} p_k^\Delta |A(p^c)^{-1}(A^k x^* - b^k)| \\ &= x^* + \sum_{j=1}^{\infty} M^j \sum_{k=1}^{K} p_k^\Delta |A(p^c)^{-1}(A^k x^* - b^k)|,\end{aligned}$$

and

$$\begin{aligned}\overline{x}' &= x^* + (I - |Y| - Z)^{-1}(y + z) \\ &= x^* + \sum_{j=1}^{\infty} (|Y| + Z)^j (y + z).\end{aligned}$$

From

$$y + z \leq \sum_{k=1}^{K} p_k^\Delta |A(p^c)^{-1}(A^k x^* - b^k)|$$

and $0 \leq |Y| + Z \leq M$,we obtain $\overline{x}' \leq \overline{x}$. We proceed Similarly for $\underline{x}' \geq \underline{x}$. ∎

4.2. Refinement of the Hansen–Bliek–Rohn bounds.

We will refine the Hansen–Bliek–Rohn bounds in the same manner as the Bauer–Skeel ones. If $\underline{a}^k \geq 0$, then

$$\begin{aligned}&|A(p^c)^{-1}(A^k x - b^k)| \\ &\qquad = A(p^c)^{-1} A^k x - A(p^c)^{-1} b^k. \quad (13)\end{aligned}$$

Otherwise, if $\overline{a}^k \leq 0$, then

$$\begin{aligned}&|A(p^c)^{-1}(A^k x - b^k)| \\ &\qquad = -A(p^c)^{-1} A^k x + A(p^c)^{-1} b^k. \quad (14)\end{aligned}$$

Otherwise, we use the standard estimation for the Hansen–Bliek–Rohn method,

$$\begin{aligned}&|A(p^c)^{-1}(A^k x - b^k)| \\ &\qquad \leq |A(p^c)^{-1} A^k||x| + |A(p^c)^{-1} b^k|. \quad (15)\end{aligned}$$

Thus (7) takes the form of

$$\begin{aligned}|x - x^*| &\leq \sum_{k=1}^{K} p_k^\Delta |A(p^c)^{-1}(A^k x - b^k)| \\ &\leq Yx - y + Z|x| + z \\ &\leq (|Y| + Z)|x| - y + z,\end{aligned}$$

where $Y, Z \in \mathbb{R}^{n \times n}$, $y, z \in \mathbb{R}^n$, and $Z \geq 0$. Next, we proceed as in the proof of Theorem 5. The method is summarized in Algorithm 2.

We show that the refinement of the Hansen–Bliek–Rohn method is in each component at least as tight as the original Hansen–Bliek–Rohn bounds.

Proposition 4. *Let $\boldsymbol{x}$ be the enclosure obtained by Theorem 5, and $\boldsymbol{x}'$ the enclosure obtained by Algorithm 2. Then $\boldsymbol{x}' \subseteq \boldsymbol{x}$.*

Algorithm 2 (Refinement of the Hansen–Bliek–Rohn method)

1: $Y := 0$; $y := 0$; $Z := 0$; $z := 0$;
2: $x^* := A(p^c)^{-1}b(p^c)$;
3: Let $\boldsymbol{x}$ be an initial an enclosure to Σ;
4: **for** $k = 1, \dots, K$ **do**
5: $\quad \boldsymbol{a}^k := A(p^c)^{-1}(A^k\boldsymbol{x} - b^k)$;
6: $\quad$ **for** $j = 1, \dots, n$ **do**
7: $\quad\quad$ **if** $\underline{a}_j^k \geq 0$ **then**
8: $\quad\quad\quad Y_{j\bullet} := Y_{j\bullet} + p_k^\Delta A(p^c)^{-1}_{j\bullet}A^k;\ y_j := y_j + p_k^\Delta A(p^c)^{-1}_{j\bullet}b^k$;
9: $\quad\quad$ **else if** $\overline{a}_j^k \leq 0$ **then**
10: $\quad\quad\quad Y_{j\bullet} := Y_{j\bullet} - p_k^\Delta A(p^c)^{-1}_{j\bullet}A^k;\ y_j := y_j - p_k^\Delta A(p^c)^{-1}_{j\bullet}b^k$;
11: $\quad\quad$ **else**
12: $\quad\quad\quad Z_{j\bullet} := Z_{j\bullet} + p_k^\Delta |A(p^c)^{-1}_{j\bullet}A^k|;\ z_j := z_j + p_k^\Delta |A(p^c)^{-1}_{j\bullet}b^k|$;
13: $\quad\quad$ **end if**
14: $\quad$ **end for**
15: **end for**
16: $M^* := (I - |Y| - Z)^{-1};\ x^0 := M^*(|x^*| - y + z)$;
17: **for** $i = 1, \dots, n$ **do**
18: $\quad \overline{x}_i' := \max\left\{x_i^0 + (x_i^* - |x_i^*|)m_{ii}^*,\ \frac{1}{2m_{ii}^*-1}\big(x_i^0 + (x_i^* - |x_i^*|)m_{ii}^*\big)\right\}$;
19: $\quad \underline{x}_i' := \min\left\{-x_i^0 + (x_i^* + |x_i^*|)m_{ii}^*,\ \frac{1}{2m_{ii}^*-1}\big(-x_i^0 + (x_i^* + |x_i^*|)m_{ii}^*\big)\right\}$;
20: **end for**
21: **return** $\boldsymbol{x}'$, an enclosure to Σ.

Proof. Let $i \in \{1, \dots, n\}$. We prove $\overline{x}_i' \leq \overline{x}_i$. The lower case is done accordingly. Write

$$M'^* := (I - |Y| - Z)^{-1},$$
$$x'^0 := M^*(|x^*| - y + z).$$

Clearly, $M'^* \leq M^*$ and $x'^0 \leq x^0$. Thus

$$-x_i'^0 + (x_i^* + |x_i^*|)m_{ii}'^* \leq -x_i^0 + (x_i^* + |x_i^*|)m_{ii}^*.$$

Since $m_{ii}'^* \geq 1$, we have

$$\frac{1}{2m_{ii}'^* - 1} \leq 1,$$

and the term

$$\frac{1}{2m_{ii}^* - 1}\big(x_i^0 + (x_i^* - |x_i^*|)m_{ii}^*\big)$$

is the maximizer in Step 18 of Algorithm 2 if and only if it is non-positive. In this case,

$$\frac{1}{2m_{ii}'^* - 1}\big(x_i'^0 + (x_i^* - |x_i^*|)m_{ii}'^*\big) \leq \frac{1}{2m_{ii}^* - 1}\big(x_i^0 + (x_i^* - |x_i^*|)m_{ii}^*\big),$$

which completes the proof. ■

5. Time complexity

Let us analyse the theoretical time complexity of the proposed methods. Both Bauer–Skeel and Hansen–Bliek–Rohn methods have the same asymptotic time complexities. The most computationally expensive is to calculate the matrix M. It costs $\mathcal{O}(n^3K)$ operations by using a naive implementation. However, the matrices A^k, $k = 1, \dots, K$, are usually sparse, in which case the complexity is lower.

Denote by P the maximum number of non-zero entries in some A^k, $k = 1, \dots, K$, that is, the maximum number of appearances of some parameter p_k. Then, computation of M can be implemented in $\mathcal{O}(nK(n+P))$, the matrix inverse is in $\mathcal{O}(n^3)$ and the remaining calculation is negligible with respect to the worst case time complexity. Thus the algorithms are in $\mathcal{O}(n^3 + n^2K + nPK)$.

For instance, for symmetric interval systems, we have $P = 2$, $K = \frac{1}{2}n(n-1)$, so the total cost is $\mathcal{O}(n^4)$. For Toeplitz systems we have $P = \mathcal{O}(n)$, $K = \mathcal{O}(n)$, so the time complexity is $\mathcal{O}(n^3)$.

Concerning the refinements discussed in Section 4 it turns out that their asymptotic time complexity is the same as that of the original methods, that is, $\mathcal{O}(n^3 + n^2K + nPK)$. Of course, the multiplicative terms are greater, which causes the higher computational time presented in Section 6.

The iterative methods by Rump or Popova and Krämer require $\mathcal{O}(n^3 + n^2KI)$ operations, where I stands for the number of iterations. Thus our approach is not asymptotically worse provided that $P = \mathcal{O}(nI)$.

6. Examples and numerical experiments

In his paper, Rohn (2010) claims that for the standard system of interval linear equations the Hansen–Bliek–Rohn bounds are never worse than the Bauer–Skeel ones. In the following examples we show that this is not the case for (more general) parametric systems. Surprisingly, the Bauer–Skeel bounds are sometimes notably better (Example 2).

Example 1. Consider Okumura's problem of a linear resistive network (Popova and Krämer, 2008, Example 5.3.). It obeys the form of (1) with

$$A(p) = \begin{pmatrix} p_1+p_6 & -p_6 & 0 & 0 & 0 \\ -p_6 & p_2+p_6+p_7 & -p_7 & 0 & 0 \\ 0 & -p_7 & p_3+p_7+p_8 & -p_8 & 0 \\ 0 & 0 & -p_8 & p_4+p_8+p_9 & -p_9 \\ 0 & 0 & 0 & -p_9 & p_5+p_9 \end{pmatrix},$$

$b(p) = (10, 0, 10, 0, 0)^T$, and $p_i \in [0.99, 1.01]$, $i = 1, \ldots, 9$. The Bauer–Skeel bounds computed according to Theorem 4 are

$$([7.0148, 7.1671], [4.1173, 4.2463], [5.3933, 5.5158], [2.1377, 2.2260], [1.0601, 1.1217])^T,$$

and the refinement by Algorithm 1 yields

$$([7.0151, 7.1667], [4.1180, 4.2456], [5.3938, 5.5153], [2.1382, 2.2255], [1.0605, 1.1213])^T.$$

The Hansen–Bliek–Rohn method (Theorem 5) results in a not-as-tight enclosure,

$$([6.9693, 7.2150], [4.0689, 4.2971], [5.3501, 5.5612], [2.1083, 2.2568], [1.0397, 1.1431])^T.$$

The refinement by Algorithm 2 gives

$$([6.9925, 7.1913], [4.1134, 4.2504], [5.3799, 5.5307], [2.1324, 2.2317], [1.0576, 1.1244])^T.$$

Notice that for this example the exact interval hull of the solution set Σ is known (Popova and Krämer, 2008),

$$([7.0170, 7.1663], [4.1193, 4.2454], [5.3952, 5.5150], [2.1392, 2.2253], [1.0614, 1.1211])^T.$$

♦

Example 2. From Example 3.4 of Popova and Krämer (2008) we have

$$\begin{pmatrix} p_1 & p_2-1 \\ p_2 & p_1 \end{pmatrix} x = \begin{pmatrix} -p_2+\frac{1}{3} \\ p_2 \end{pmatrix},$$

where $p_1 \in [-2, -1]$ and $p_2 \in [3, 5]$. Here, the Bauer–Skeel enclosure gives

$$([0.1282, 1.2052], [-1.4103, -0.3675])^T,$$

whereas the Hansen–Bliek–Rohn method yields

$$([-0.4359, 3.7693], [-4.8718, -0.0923])^T.$$

No refinement for this very low dimensional example was successful. ♦

Example 3. The last example is devoted to numerical experiments with randomly generated data. Even though the real-life data are not random, such experiments reveal something on the performance of algorithms. The computations were carried out in MATLAB 7.7.0.471 (R2008b) on a machine with an `AMD Athlon 64 X2 Dual Core Processor 4400+`, 2.2 GHz, CPU with 1004 MB RAM. Interval arithmetics and some basic interval functions were provided by the interval toolbox INTLAB v5.3 (Rump, 2006). We used just a simple implementation of the methods presented. Notice, for large-scale problems in particular, that a more subtle implementation utilizing the sparsity of matrices A^k, $k = 1, \ldots, K$, could be used.

First, we consider systems with symmetric matrices that were generated in the following way. First, entries of A^c were chosen randomly and independently in $[-10, 10]$ with uniform distribution, and then we set $A^c := A^c + (A^c)^T + 10nI$. The entries of the radius matrix A^Δ are equal to R, where $R > 0$ is a parameter. The right-hand side interval vector was chosen to be degenerate (zero width) with entries chosen randomly from $[-10, 10]$.

In diverse settings of the dimension n and the radius R we carried out sequences of 10 runs. The results are summarized in Table 1. We compare the resulting enclosures by relative sums of radii with respect to the Bauer–Skeel bounds. That is, for a given enclosure $\boldsymbol{w}$ and the Bauer–Skeel enclosure $\boldsymbol{v}$, we display

$$\frac{\sum_{i=1}^{n} w_i^\Delta}{\sum_{i=1}^{n} v_i^\Delta}.$$

On the average, the Bauer–Skeel (BS) method gives tighter enclosures than the Hansen–Bliek–Rohn (HBR) one. The refinement is more conclusive for the latter than for the former.

Table 1. Symmetric systems with random data.

n	R	relative sum of radii				average execution time (in sec.)			
		BS	refined BS	HBR	refined HBR	BS	refined BS	HBR	refined HBR
5	0.05	1	0.9994	1.06	1.003	0.01537	0.0893	0.01421	0.0886
5	0.1	1	0.9988	1.058	1.001	0.0155	0.09176	0.0148	0.08898
5	0.5	1	0.9947	1.044	0.9931	0.01553	0.09054	0.01441	0.0911
5	1	1	0.9907	1.026	0.9795	0.01412	0.08438	0.01366	0.08453
10	0.05	1	0.999	1.1	1.001	0.04731	0.5632	0.0456	0.5579
10	0.1	1	0.9981	1.099	1.001	0.04588	0.5559	0.04498	0.5494
10	0.5	1	0.9912	1.092	1.001	0.04839	0.5813	0.04604	0.5679
10	1	1	0.984	1.082	1	0.04638	0.5461	0.04401	0.5449
15	0.05	1	0.999	1.104	1	0.1017	1.802	0.1	1.778
15	0.1	1	0.9979	1.103	1	0.09783	1.719	0.09587	1.695
15	0.5	1	0.9903	1.099	1.001	0.09836	1.759	0.09593	1.724
15	1	1	0.9825	1.092	1.004	0.09666	1.733	0.0956	1.731
20	0.05	1	0.999	1.104	1.001	0.1758	3.979	0.1695	3.956
20	0.1	1	0.998	1.103	1.001	0.1721	3.937	0.1697	3.928
20	0.5	1	0.9906	1.1	1.003	0.1726	3.961	0.1671	3.976
20	1	1	0.9831	1.095	1.008	0.1699	4.01	0.169	3.996
25	0.05	1	0.999	1.097	1	0.2774	7.591	0.2647	7.524
25	0.1	1	0.9981	1.096	1	0.283	7.712	0.2775	7.644
25	0.5	1	0.9909	1.094	1.003	0.2726	7.599	0.2669	7.493
25	1	1	0.9837	1.09	1.008	0.2767	7.723	0.2704	7.77
50	0.05	1	0.999	1.099	1	6.399	57.36	6.327	56.72
50	0.1	1	0.9981	1.099	1.001	6.505	57.13	6.31	56.63
50	0.5	1	0.9911	1.097	1.006	6.395	57.64	6.341	57.11
50	1	1	0.984	1.095	1.013	6.371	57.78	6.317	57.36
100	0.05	1	0.999	1.095	1	90.71	511.9	90.07	488.2
100	0.1	1	0.9981	1.095	1.001	91.75	527	88.77	489.9
100	0.5	1	0.991	1.095	1.006	92.68	526.7	89.01	489.1
100	1	1	0.9838	1.094	1.014	90.5	522.4	89.23	498.7

Second, we consider Toeplitz systems, i.e, systems with matrices A satisfying $a_{i,j} = a_{i+1,j+1}$, $i,j = 1,\dots,n-1$. Herein, $A^c_{i,1}$ and $A^c_{1,i}$, $i = 2,\dots,n$, were chosen randomly in $[-10,10]$, whereas $A^c_{1,1}$ in $[-10+10n, 10+10n]$. The entries of A^Δ are equal to R. The right-hand side vector was again degenerate with entries selected randomly in $[-10,10]$. The results are displayed in Table 2.

Third, we consider symmetric systems again generated in the same way as above. We compare the combination of the Bauer–Skeel and Hansen–Bliek–Rohn methods (Remark 1) with the interval Cholesky method (Alefeld and Mayer, 1993; 2008). We implemented the basic version of the interval Cholesky method since the more sophisticated algorithm based on pivot tightening (Garloff, 2010) is intractable, having the exponential complexity. Table 3 demonstrates that the proposed method is much more efficient than the interval Cholesky one. Even though the computing time is slightly better for the latter, the former yields a significantly tighter enclosure.

Finally, we did some comparisons with the parametric solver by Popova (2004a; 2006b); see Table 4. Again, we considered symmetric interval systems. On the average, our approach is slightly better, and the refinement is more significantly better.

7. Concluding remarks

Numerical experiments revealed that a generalization of the Bauer–Skeel method is a competitive alternative to the Hansen–Bliek–Rohn method. It is best to use a combination of both to obtain a tight enclosure. As observed in the numerical experiments, the resulting (direct) algorithm is a competitive alternative to existing direct or iterative algorithms. Moreover, efficient refinements of both methods were proposed in order to compute tighter enclosures.

As pointed out by one referee, the performance of this centered form approach is limited (cf. Neumaier and Pownuk, 2007). A non-centered form approach may lead to further improvements.

Acknowledgment

This paper was partially supported by the Czech Science Foundation Grant P403/12/1947.

Table 2. Toeplitz systems with random data.

n	R	relative sum of radii				average execution time (in sec.)			
		BS	refined BS	HBR	refined HBR	BS	refined BS	HBR	refined HBR
5	0.05	1	0.9985	1.217	1.01	0.008363	0.05246	0.008132	0.05139
5	0.1	1	0.997	1.215	1.008	0.0102	0.05672	0.009942	0.05684
5	0.5	1	0.9859	1.2	1.01	0.01007	0.05786	0.009805	0.05706
5	1	1	0.976	1.179	1.014	0.01092	0.05661	0.009788	0.05831
10	0.05	1	0.9979	1.317	1.002	0.01879	0.2069	0.01853	0.21
10	0.1	1	0.9959	1.316	1.001	0.01832	0.2046	0.0172	0.2001
10	0.5	1	0.9792	1.307	0.9978	0.018	0.2023	0.01751	0.1963
10	1	1	0.9588	1.295	0.9967	0.01894	0.2084	0.01806	0.2091
15	0.05	1	0.9979	1.363	1.005	0.02552	0.4194	0.02465	0.4127
15	0.1	1	0.9958	1.362	1.005	0.02636	0.4542	0.03089	0.455
15	0.5	1	0.9794	1.356	1.012	0.02747	0.4478	0.02632	0.4443
15	1	1	0.9605	1.349	1.03	0.02684	0.4291	0.02627	0.4324
20	0.05	1	0.9978	1.389	1.008	0.03518	0.7715	0.03548	0.7492
20	0.1	1	0.9956	1.388	1.01	0.03628	0.7911	0.03554	0.7797
20	0.5	1	0.9786	1.384	1.024	0.03488	0.757	0.03406	0.7627
20	1	1	0.9585	1.378	1.038	0.03498	0.769	0.03447	0.776
25	0.05	1	0.9977	1.421	1.007	0.04478	1.157	0.04359	1.156
25	0.1	1	0.9954	1.42	1.009	0.04647	1.195	0.04478	1.192
25	0.5	1	0.9779	1.417	1.031	0.0467	1.198	0.04404	1.194
25	1	1	0.9582	1.412	1.061	0.04455	1.166	0.04308	1.165
50	0.05	1	0.9978	1.418	1.005	0.5689	4.549	0.5276	4.48
50	0.1	1	0.9956	1.418	1.009	0.528	4.519	0.526	4.509
50	0.5	1	0.9787	1.416	1.035	0.5322	4.719	0.535	4.629
50	1	1	0.9599	1.414	1.068	0.5278	4.634	0.531	4.616
100	0.05	1	0.9976	1.452	1.004	3.704	20.19	3.694	19.7
100	0.1	1	0.9953	1.452	1.008	3.717	20.13	3.91	19.84
100	0.5	1	0.9776	1.451	1.043	3.719	20.22	3.705	20.11
100	1	1	0.9582	1.45	1.087	3.678	20.41	3.663	20.26

References

Alefeld, G., Kreinovich, V. and Mayer, G. (1997). On the shape of the symmetric, persymmetric, and skew-symmetric solution set, *SIAM Journal on Matrix Analysis and Applications* **18**(3): 693–705.

Alefeld, G., Kreinovich, V. and Mayer, G. (2003). On the solution sets of particular classes of linear interval systems, *Journal of Computational and Applied Mathematics* **152**(1–2): 1–15.

Alefeld, G. and Mayer, G. (1993). The Cholesky method for interval data, *Linear Algebra and Its Applications* **194**: 161–182.

Alefeld, G. and Mayer, G. (2008). New criteria for the feasibility of the Cholesky method with interval data, *SIAM Journal on Matrix Analysis and Applications* **30**(4): 1392–1405.

Beeck, H. (1975). Zur Problematik der Hüllenbestimmung von Intervallgleichungssystem en, *in* K. Nickel (Ed.), *Interval Mathematics: Proceedings of the International Symposium on Interval Mathematics*, Lecture Notes in Computer Science, Vol. 29, Springer, Berlin, pp. 150–159.

Busłowicz, M. (2010). Robust stability of positive continuous-time linear systems with delays, *International Journal of Applied Mathematics and Computer Science* **20**(4): 665–670, DOI: 10.2478/v10006-010-0049-8.

Fiedler, M., Nedoma, J., Ramík, J., Rohn, J. and Zimmermann, K. (2006). *Linear Optimization Problems with Inexact Data*, Springer, New York, NY.

Garloff, J. (2010). Pivot tightening for the interval Cholesky method, *Proceedings in Applied Mathematics and Mechanics* **10**(1): 549–550.

Hladík, M. (2008). Description of symmetric and skew-symmetric solution set, *SIAM Journal on Matrix Analysis and Applications* **30**(2): 509–521.

Horn, R.A. and Johnson, C.R. (1985). *Matrix Analysis*, Cambridge University Press, Cambridge.

Jansson, C. (1991). Interval linear systems with symmetric matrices, skew-symmetric matrices and dependencies in the right hand side, *Computing* **46**(3): 265–274.

Kolev, L.V. (2004). A method for outer interval solution of linear parametric systems, *Reliable Computing* **10**(3): 227–239.

Kolev, L.V. (2006). Improvement of a direct method for outer solution of linear parametric systems, *Reliable Computing* **12**(3): 193–202.

Table 3. Comparison with the interval Cholesky method.

n	R	relative sum of radii		average exec. time (in sec.)	
		HBR–BS	Cholesky	HBR–BS	Cholesky
5	0.05	1	187.0	0.01639	0.009779
5	0.1	1	92.69	0.01649	0.009768
5	0.5	1	17.29	0.01637	0.009774
5	1	1	7.891	0.01643	0.009801
10	0.05	1	200.2	0.05505	0.009791
10	0.1	1	99.19	0.05527	0.009797
10	0.5	1	18.4	0.05724	0.009793
10	1	1	8.298	0.05544	0.009818
15	0.05	1	238.7	0.1195	0.00984
15	0.1	1	118.3	0.1199	0.009813
15	0.5	1	21.94	0.1202	0.009844
15	1	1	9.9	0.1202	0.009806
20	0.05	1	232.6	0.2118	0.01084
20	0.1	1	115.3	0.2111	0.0102
20	0.5	1	21.42	0.2105	0.009848
20	1	1	9.685	0.2112	0.00987
25	0.05	1	235.9	0.3365	0.009854
25	0.1	1	116.9	0.3361	0.009834
25	0.5	1	21.76	0.3365	0.009831
25	1	1	9.866	0.3361	0.009834

Merlet, J.-P. (2009). Interval analysis for certified numerical solution of problems in robotics, *International Journal of Applied Mathematics and Computer Science* **19**(3): 399–412, DOI: 10.2478/v10006-009-0033-3.

Meyer, C.D. (2000). *Matrix Analysis and Applied Linear Algebra*, SIAM, Philadelphia, PA.

Neumaier, A. (1990). *Interval Methods for Systems of Equations*, Cambridge University Press, Cambridge.

Neumaier, A. (1999). A simple derivation of the Hansen–Bliek–Rohn–Ning–Kearfott enclosure for linear interval equations, *Reliable Computing* **5**(2): 131–136.

Neumaier, A. and Pownuk, A. (2007). Linear systems with large uncertainties, with applications to truss structures, *Reliable Computing* **13**(2): 149–172.

Ning, S. and Kearfott, R.B. (1997). A comparison of some methods for solving linear interval equations, *SIAM Journal on Numerical Analysis* **34**(4): 1289–1305.

Padberg, M. (1999). *Linear Optimization and Extensions,* 2nd Edn., Springer, Berlin.

Popova, E. (2002). Quality of the solution sets of parameter-dependent interval linear systems, *Zeitschrift für Angewandte Mathematik und Mechanik* **82**(10): 723–727.

Popova, E.D. (2001). On the solution of parametrised linear systems, *in* W. Krämer and J.W. von Gudenberg (Eds.), *Scientific Computing, Validated Numerics, Interval Methods*, Kluwer, London, pp. 127–138.

Popova, E.D. (2004a). Parametric interval linear solver, *Numerical Algorithms* **37**(1–4): 345–356.

Popova, E.D. (2004b). Strong regularity of parametric interval matrices, *in* I. Dimovski (Ed.), *Mathematics and Education in Mathematics, Proceedings of the 33rd Spring Conference of the Union of Bulgarian Mathematicians, Borovets, Bulgaria*, BAS, Sofia, pp. 446–451.

Popova, E.D. (2006a). Computer-assisted proofs in solving linear parametric problems, *12th GAMM/IMACS International Symposium on Scientific Computing, Computer Arithmetic and Validated Numerics, SCAN 2006, Duisburg, Germany*, p. 35.

Popova, E.D. (2006b). Webcomputing service framework, *International Journal Information Theories & Applications* **13**(3): 246–254.

Popova, E.D. (2009). Explicit characterization of a class of parametric solution sets, *Comptes Rendus de L'Academie Bulgare des Sciences* **62**(10): 1207–1216.

Popova, E.D. and Krämer, W. (2007). Inner and outer bounds for the solution set of parametric linear systems, *Journal of Computational and Applied Mathematics* **199**(2): 310–316.

Popova, E.D. and Krämer, W. (2008). Visualizing parametric solution sets, *BIT Numerical Mathematics* **48**(1): 95–115.

Rex, G. and Rohn, J. (1998). Sufficient conditions for regularity and singularity of interval matrices, *SIAM Journal on Matrix Analysis and Applications* **20**(2): 437–445.

Rohn, J. (1989). Systems of linear interval equations, *Linear Algebra and Its Applications* **126**(C): 39–78.

Rohn, J. (1993). Cheap and tight bounds: The recent result by E. Hansen can be made more efficient, *Interval Computations* (4): 13–21.

Rohn, J. (2004). A method for handling dependent data in interval linear systems, *Technical Report*

Table 4. Comparison with the parametric solver by Popova.

n	R	relative sum of radii		
		HBR-BS	refined HBR-BS	parametric solver
5	0.1	1	0.9945	0.9991
5	0.5	1	0.9773	1.0046
5	1	1	0.9674	1.0191
10	0.1	1	0.9970	1.0001
10	0.5	1	0.9879	0.9996
10	1	1	0.9799	1.0000
15	0.1	1	0.9971	0.9977
15	0.5	1	0.9883	1.0002
15	1	1	0.9814	1.0097
20	0.1	1	0.9971	1.0015
20	0.5	1	0.9881	1.0002
20	1	1	0.9783	1.0000
25	0.1	1	0.9975	1.0018
25	0.5	1	0.9874	1.0002
25	1	1	0.9792	1.0005

911, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, `http://uivtx.cs.cas.cz/~rohn/publist/rp911.ps`.

Rohn, J. (2010). An improvement of the Bauer–Skeel bounds, *Technical Report V-1065*, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, `http://uivtx.cs.cas.cz/~rohn/publist/bauerskeel.pdf`.

Rump, S.M. (1983). Solving algebraic problems with high accuracy, *in* U. Kulisch and W. Miranker (Eds.), *A New Approach to Scientific Computation*, Academic Press, New York, NY, pp. 51–120.

Rump, S.M. (1994). Verification methods for dense and sparse systems of equations, *in* J. Herzberger (Ed.), *Topics in Validated Computations*, Studies in Computational Mathematics, Elsevier, Amsterdam, pp. 63–136.

Rump, S.M. (2006). INTLAB—Interval Laboratory, the Matlab toolbox for verified computations, Version 5.3. `http://www.ti3.tu-harburg.de/rump/intlab/`.

Rump, S.M. (2010). Verification methods: Rigorous results using floating-point arithmetic, *Acta Numerica* **19**: 287–449.

Schrijver, A. (1998). *Theory of Linear and Integer Programming,* Reprint Edn., Wiley, Chichester.

Skalna, I. (2006). A method for outer interval solution of systems of linear equations depending linearly on interval parameters, *Reliable Computing* **12**(2): 107–120.

Skalna, I. (2008). On checking the monotonicity of parametric interval solution of linear structural systems, *in* R. Wyrzykowski, J. Dangarra, K. Karczewski and J. Wasniewski (Eds.), *Parallel Processing and Applied Mathematics*, Lecture Notes in Computer Science, Vol. 4967, Springer-Verlag, Berlin/Heidelberg, pp. 1400–1409.

Stewart, G. W. (1998). *Matrix Algorithms, Vol. 1: Basic Decompositions*, SIAM, Philadelphia, PA.

Milan Hladík received his Ph.D. degree in operations research from Charles University in Prague in 2006. In 2008, he worked in the Coprin team at INRIA, Sophia Antipolis, France, as a postdoc research fellow. Since 2009, he has been an assistant professor at the Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University in Prague. His research interests are interval analysis, parametric programming and uncertain optimization.

Appendix

Consider a system of interval linear equations $\boldsymbol{A}x = \boldsymbol{b}$, which is a special case of (1), and the solution set $\hat{\Sigma} := \{x \in \mathbb{R}^n \mid Ax = b,\ A \in \boldsymbol{A},\ b \in \boldsymbol{b}\}$. The Bauer–Skeel bounds (Rohn, 2010; Stewart, 1998) and the Hansen–Bliek–Rohn bounds (Fiedler *et al.*, 2006; Rohn, 1993, Theorem 2.39) on $\hat{\Sigma}$ are given below.

Theorem 6. (Bauer–Skeel) *Let A^c nonsingular and $\rho(|(A^c)^{-1}|A^\Delta) < 1$. Write $\hat{x}^* := (A^c)^{-1}b^c$, $\hat{M} := (A^c)^{-1}|A^\Delta$ and $\hat{M}^* := (I-\hat{M})^{-1}$. Then for each $x \in \hat{\Sigma}$ we have*

$$x \geq \hat{x}^* - \hat{M}^*|(A^c)^{-1}|(A^\Delta|\hat{x}^*| + b^\Delta),$$
$$x \leq \hat{x}^* + \hat{M}^*|(A^c)^{-1}|(A^\Delta|\hat{x}^*| + b^\Delta).$$

Theorem 7. (Hansen–Bliek–Rohn) *Under the same assumption and notation as in the previous theorem, we have*

$$x_i \leq \max\left\{\hat{x}_i^0 + (\hat{x}_i^* - |\hat{x}_i^*|)\hat{m}_{ii}^*, \frac{1}{2\hat{m}_{ii}^* - 1}\left(\hat{x}_i^0 + (\hat{x}_i^* - |\hat{x}_i^*|)\hat{m}_{ii}^*\right)\right\},$$

and

$$x_i \geq \min\Big\{ -\hat{x}_i^0 + (\hat{x}_i^* + |\hat{x}_i^*|)\hat{m}_{ii}^*,$$
$$\frac{1}{2\hat{m}_{ii}^* - 1}\big(-\hat{x}_i^0 + (\hat{x}_i^* + |\hat{x}_i^*|)\hat{m}_{ii}^*\big)\Big\},$$

where $\hat{x}^0 := \hat{M}^*(|\hat{x}^*| + |(A^c)^{-1}|b^{\Delta})$.

Received: 11 July 2011
Revised: 24 November 2011