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INDEPENDENCE OF ASYMPTOTIC STABILITY OF POSITIVE 2D LINEAR SYSTEMS WITH DELAYS OF THEIR DELAYS

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It is shown that the asymptotic stability of positive 2D linear systems with delays is independent of the number and values of the delays and it depends only on the sum of the system matrices, and that the checking of the asymptotic stability of positive 2D linear systems with delays can be reduced to testing that of the corresponding positive 1D systems without delays. The effectiveness of the proposed approaches is demonstrated on numerical examples.

Keywords: 2D systems, systems with delays, asymptotic stability, positive systems.

1. Introduction

The most popular models of two-dimensional (2D) linear systems are those introduced by Roesser (1975), Fornasini and Marchesini (1976; 1997) and Kurek (1975). The models have been extended for positive systems in (Kaczorek, 1996; 2005; 2001; Valchaer, 1997). An overview of 2D linear systems theory is given in (Bose, 1982; 1985; Gałkowski, 1997; 2001; Kaczorek, 1985), and some recent results in positive systems are given in the monographs (Farina and Rinaldi, 2000; Kaczorek, 2001). Reachability and minimum energy control of positive 2D systems with one delay in states were considered in (Kaczorek, 2005). The choice of Lyapunov functions for a positive 2D Roesser model was investigated in (Kaczorek, 2007).

The notion of an internally positive 2D system (model) with delays in states and in inputs was introduced along with necessary and sufficient conditions for internal positivity, reachability, controllability, observability and the minimum energy control problem in (Kaczorek, 2002; 2006b).

The realization problem for 1D positive discrete-time systems with delays was analyzed in (Kaczorek, 2006a; 2003) and for 2D positive systems in (Kaczorek, 2004).

Internal stability and asymptotic behavior of 2D positive systems were investigated in (Valcher, 1997).

The stability of 2D positive systems described by the Roesser model and the synthesis of state-feedback controllers were considered in the paper (Hmamed *et al.*, 2008). Asymptotic stability of positive 2D linear systems was investigated in (Kaczorek, 2009a; 2009b; 2009c; 2008a; 2008c) and robust stability of positive 1D linear systems in (Busłowicz, 2007; 2008a).

In this paper it will be shown that the asymptotic stability of positive 2D linear systems with delays is independent of the number and values of the delays (it depends only on the sum of the system matrices) and the checking of the asymptotic stability of the systems with delays can be reduced to testing the stability of the corresponding 1D system without delays.

The paper is organized as follows: In Section 2 basic definitions and theorems concerning positive 2D linear systems without delays are recalled. Asymptotic stability of positive 2D linear systems without delays is addressed in Section 3. The reduction of positive 2D systems with delays to the equivalent positive 2D system without delays is considered in Section 4. The main result of the paper is presented in Section 5. It is shown that the checking of asymptotic stability of positive 2D linear systems with delays can be reduced to testing asymptotic stability of the corresponding positive 1D systems without delays. Concluding remarks are given in Section 6.

The following notation will be used: $\mathbb{R}^{n\times n}$ denotes the set of $n \times m$ real matrices, the set of real $n \times m$ matrices with nonnegative entries will be denoted by $\mathbb{R}_+^{n\times n}$ and the set of nonnegative integers by $\mathbb{Z}_+$; the $n \times n$ identity matrix will be denoted by I_n.

2. Positive 2D systems

Consider the general model of 2D linear systems:

$$\begin{aligned} x_{i+1,j+1} &= A_0x_{i,j} + A_1x_{i+1,j} + A_2x_{i,j+1} \\ &\quad + B_0u_{i,j} + B_1u_{i+1,j} + B_2u_{i,j+1}, \qquad (1a) \\ y_{i,j} &= Cx_{i,j} + Du_{i,j}, \qquad (1b) \end{aligned}$$

where $x_{i,j} \in \mathbb{R}^n$, $u_{i,j} \in \mathbb{R}^m$, $y_{i,j} \in \mathbb{R}^p$ are the state, input and output vectors at the point (i,j) and $A_k \in \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times m}$, $k = 0,1,2$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

Boundary conditions for (1a) have the form

$$x_{i,0} \in \mathbb{R}^n,\ i \in \mathbb{Z}_+ \quad \text{and} \quad x_{0,j} \in \mathbb{R}^n,\ j \in \mathbb{Z}_+. \qquad (2)$$

The model (1) is called (internally) positive if $x_{i,j} \in \mathbb{R}^n_+$ and $y_{i,j} \in \mathbb{R}^p_+$, $i,j \in \mathbb{Z}_+$ for all boundary conditions $x_{i,0} \in \mathbb{R}^n_+$, $i \in \mathbb{Z}_+$, $x_{0,j} \in \mathbb{R}^n_+$, $j \in \mathbb{Z}_+$ and every input sequence $u_{i,j} \in \mathbb{R}^m_+$, $i,j \in \mathbb{Z}_+$.

Theorem 1. (Kaczorek, 2001) *The system (1) is positive if and only if*

$$\begin{aligned} &A_k \in \mathbb{R}^{n\times n}_+, \quad B_k \in \mathbb{R}^{n\times m}_+, \quad k = 0,1,2, \\ &C \in \mathbb{R}^{p\times n}_+, \quad D \in \mathbb{R}^{p\times m}_+. \end{aligned} \qquad (3)$$

Substituting $B_1 = B_2 = 0$ and $B_0 = B$ in (1a), we obtain the first Fornasini-Marchesini model (FF-MM), and substituting $A_0 = 0$ and $B_0 = 0$ in (1a), we obtain the second Fornasini-Marchesini model (SF-MM).

The Roesser model of 2D linear systems has the form

$$\begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h_{i,j} \\ x^v_{i,j} \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix} u_{i,j}, \qquad (4a)$$

$$y_{i,j} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x^h_{i,j} \\ x^v_{i,j} \end{bmatrix} + Du_{i,j}, \qquad (4b)$$

$i,j \in \mathbb{Z}_+$, where $x^h_{i,j} \in \mathbb{R}^{n_1}$ and $x^v_{i,j} \in \mathbb{R}^{n_2}$ are the horizontal and vertical state vectors at the point (i,j), $u_{i,j} \in \mathbb{R}^m$ and $y_{i,j} \in R^p$ are the input and output vectors and $A_{kl} \in \mathbb{R}^{n_k\times n_l}$, $k,l = 1,2$, $B_{11} \in \mathbb{R}^{n_1\times m}$, $B_{22} \in \mathbb{R}^{n_2\times m}$, $C_1 \in \mathbb{R}^{p\times n_1}$, $C_2 \in \mathbb{R}^{p\times n_1}$, $D \in \mathbb{R}^{p\times m}$.

Boundary conditions for (4a) have the form

$$x^h_{0,j} \in \mathbb{R}^{n_1},\ j \in \mathbb{Z}_+ \quad \textit{and} \quad x^v_{i,0} \in \mathbb{R}^{n_2},\ i \in \mathbb{Z}_+, \qquad (5)$$

The model (4) is called an (internally) positive Roesser model if $x^h_{i,j} \in \mathbb{R}^{n_1}_+$, $x^v_{i,j} \in \mathbb{R}^{n_2}_+$ and $y_{i,j} \in \mathbb{R}^p_+$, $i,j \in \mathbb{Z}_+$ for any nonnegative boundary conditions

$$x^h_{0,j} \in \mathbb{R}^{n_1}_+,\ j \in \mathbb{Z}_+, \quad x^v_{i,0} \in \mathbb{R}^{n_2}_+,\ i \in \mathbb{Z}_+ \qquad (6)$$

and all input sequences $u_{i,j} \in \mathbb{R}^m_+$, $i,j \in Z_+$.

Theorem 2. (Kaczorek, 2001) *The system Roesser model is positive if and only if*

$$\begin{gathered} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathbb{R}^{n\times n}_+, \quad \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix} \in \mathbb{R}^{n\times m}_+, \\ \begin{bmatrix} C_1 & C_2 \end{bmatrix} \in \mathbb{R}^{p\times n}_+, \qquad D \in \mathbb{R}^{p\times m}_+, \\ n = n_1 + n_2. \end{gathered} \qquad (7)$$

Defining

$$\begin{gathered} x_{i,j} = \begin{bmatrix} x^h_{i,j} \\ x^v_{i,j} \end{bmatrix}, \\ A_1 = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad A_2 = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}, \\ B_1 = \begin{bmatrix} 0 \\ B_{22} \end{bmatrix}, \qquad B_2 = \begin{bmatrix} B_{11} \\ 0 \end{bmatrix} \end{gathered} \qquad (8)$$

we may write the Roesser model in the SF-MM form

$$x_{i+1,j+1} = A_1x_{i+1,j} + A_2x_{i,j+1} + B_1u_{i+1,j} + B_2u_{i,j+1}. \qquad (9)$$

3. Asymptotic stability of positive 2D systems without delays

The positive general model (1a) is called asymptotically stable if for any bounded boundary conditions $x_{i,0} \in \mathbb{R}^n_+$, $i \in \mathbb{Z}_+$, $x_{0,j} \in \mathbb{R}^n_+$, $j \in \mathbb{Z}_+$ and zero inputs $u_{i,j} = 0$, $i,j \in \mathbb{Z}_+$,

$$\lim_{i,j\to\infty} x_{i,j} = 0 \qquad (10)$$

for all $x_{i,0} \in \mathbb{R}^n_+$, $x_{0,j} \in \mathbb{R}^n_+$, $i,j \in \mathbb{Z}_+$.

Theorem 3. (Kaczorek, 2009a; 2008a) *For the positive general model (1), the following statements are equivalent:*

1. *The positive general model (1) is asymptotically stable.*
2. *All the coefficients $\hat{a}_i$, $i = 0,1,\dots,n-1$ of the characteristic polynomial of the matrix $\hat{A} = A - I_n$, $A = A_0 + A_1 + A_2$,*

$$\begin{aligned} p_{\hat{A}}(z) &= \det[I_nz - \hat{A}] \\ &= z^n + \hat{a}_{n-1}z^{n-1} + \cdots + \hat{a}_1z + \hat{a}_0, \end{aligned} \qquad (11)$$

are positive.

3. *All principal minors of the matrix*

$$\bar{A} = I_n - A = \begin{bmatrix} \bar{a}_{11} & \dots & \bar{a}_{1n} \\ \vdots & \dots & \vdots \\ \bar{a}_{n1} & \dots & \bar{a}_{nn} \end{bmatrix} \qquad (12)$$

are positive, i.e.,

$$|\bar{a}_{11}| > 0, \quad \begin{vmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{vmatrix} > 0, \quad \dots, \quad \det[I_n - A] > 0. \tag{13}$$

Theorem 4. (Kaczorek, 2001) *The positive general model (1) is unstable if at least one diagonal entry of the matrix $A = A_0 + A_1 + A_2$ is greater than 1.*

In the particular case when $A_0 = 0$, we have the following result.

Theorem 5. *For the positive 2D SF-MM (9), the following statements are equivalent:*

1. *The positive 2D SF-MM (9) is asymptotically stable.*
2. *All coefficients a'_i, $i = 0, 1, \dots, n-1$ of the characteristic polynomial of the matrix $A' = A_1 + A_2 - I_n$,*

$$\begin{aligned} p_{A'}(z) &= \det[I_n z - A'] \\ &= z^n + a'_{n-1}z^{n-1} + \dots + a'_1 z + a'_0, \end{aligned} \tag{14}$$

are positive.

3. *All principal minors of the matrix*

$$-A' = I_n - A_1 - A_2 = \begin{bmatrix} a'_{11} & \dots & a'_{1n} \\ \vdots & \dots & \vdots \\ a'_{n1} & \dots & a'_{nn} \end{bmatrix} \tag{15}$$

are positive, i.e.,

$$|a'_{11}| > 0, \quad \begin{vmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{vmatrix} > 0, \quad \vdots \quad \det[I_n - A_1 - A_2] > 0. \tag{16}$$

Similarly, for the positive 2D Roesser model (4), we have the following result.

Theorem 6. *For the positive 2D Roesser model (4), the following statements are equivalent:*

1. *The positive 2D Roesser model (4) is asymptotically stable.*
2. *All the coefficients a_i, $i = 0, 1, \dots, n-1$ of the characteristic polynomial of the matrix*

$$\begin{aligned} &p_R(z) \\ &= \det \begin{bmatrix} I_{n_1}(z+1) - A_{11} & -A_{12} \\ -A_{21} & I_{n_2}(z+1) - A_{22} \end{bmatrix} \\ &= z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0, \end{aligned} \tag{17}$$

where $n = n_1 + n_2$, are positive.

3. *All principal minors of the matrix*

$$\begin{bmatrix} I_{n_1} - A_{11} & -A_{12} \\ -A_{21} & I_{n_2} - A_{22} \end{bmatrix}$$

are positive.

The proof follows immediately from Corollary 2 and Theorem 3 in (Kaczorek, 2009b).

Example 1. Using Theorem 3, check the asymptotic stability of the positive general 2D model (1) with the matrices

$$A_0 = \begin{bmatrix} 0.1 & 0.3 \\ 0 & 0.2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.2 & 1 \\ 0 & 0.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.3 & 1 \\ 0 & 0.4 \end{bmatrix}. \tag{18}$$

Using (11), we obtain

$$\hat{A} = A_0 + A_1 + A_2 - I_2 = \begin{bmatrix} -0.4 & 1.4 \\ 0 & -0.3 \end{bmatrix}$$

and

$$\begin{aligned} p_{\hat{A}}(z) = \det[I_2 z - \hat{A}] &= \begin{vmatrix} z+0.4 & -1.4 \\ 0 & z+0.3 \end{vmatrix} \\ &= z^2 + 0.7z + 0.12. \end{aligned}$$

In this case, the matrix (12) has the form

$$\bar{A} = -\hat{A} = \begin{bmatrix} 0.4 & -1.4 \\ 0 & 0.3 \end{bmatrix},$$

and its principal minors are positive, since $|a_{11}| = 0.4$ and $\det \bar{A} = 0.12$. The conditions (2) and (3) of Theorem 3 are satisfied and the general 2D model with (18) is asymptotically stable.

Example 2. Using Theorem 6, check the asymptotic stability of the positive 2D Roesser model (4) with the matrices

$$A_{11} = \begin{bmatrix} 0.1 & 0.2 \\ 0 & 0.3 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 & 0.1 \end{bmatrix}, \quad A_{22} = [0.4]. \tag{19}$$

Using (17), we obtain

$$\begin{aligned} p_R(z) &= \det \begin{bmatrix} I_2(z+1) - A_{11} & -A_{12} \\ -A_{21} & I_1(z+1) - A_{22} \end{bmatrix} \\ &= \begin{vmatrix} z+0.9 & -0.2 & -0.2 \\ 0 & z+0.7 & -0.1 \\ 0 & -0.1 & z+0.6 \end{vmatrix} \\ &= z^3 + 2.2z^2 + 1.58z + 0.369. \end{aligned}$$

The principal minors of the matrix

$$\begin{bmatrix} I_2 - A_{11} & -A_{12} \\ -A_{21} & I_1 - A_{22} \end{bmatrix} = \begin{bmatrix} 0.9 & -0.2 & -0.2 \\ 0 & 0.7 & -0.1 \\ 0 & -0.1 & 0.6 \end{bmatrix}$$

are positive since

$$M_1 = 0.9, \quad M_2 = \begin{vmatrix} 0.9 & -0.2 \\ 0 & 0.7 \end{vmatrix} = 0.63,$$

$$M_3 = \begin{vmatrix} 0.9 & -0.2 & -0.2 \\ 0 & 0.7 & -0.1 \\ 0 & -0.1 & 0.6 \end{vmatrix} = 0.369.$$

Therefore, the conditions (2) and (3) of Theorem 6 are satisfied and the positive 2D Roesser model with (19) is asymptotically stable.

4. Positive 2D systems with delays

4.1. 2D general model with delays. Consider the autonomous positive 2D general model with q delays in state

$$x_{i+1,j+1} = \sum_{k=0}^{q} \big(A_k^0 x_{i-k,j-k} + A_k^1 x_{i+1-k,j-k} + A_k^2 x_{i-k,j+1-k}\big), \tag{20}$$

$i,j \in \mathbb{Z}_+$, where $x_{i,j} \in \mathbb{R}_+^n$ is the state vector at the point (i,j) and $A_k^t \in \mathbb{R}_+^{n\times n}$, $k = 0,1,\dots,q$; $t = 0,1,2$.

Defining the vector

$$\bar{x}_{i,j} = \begin{bmatrix} x_{i,j} \\ x_{i-1,j-1} \\ x_{i-2,j-2} \\ \vdots \\ x_{i-q,j-q} \end{bmatrix} \in \mathbb{R}^{\bar{N}}, \quad \bar{N} = (q+1)n \tag{21a}$$

and the matrices

$$\bar{A}_0 = \begin{bmatrix} A_0^0 & A_1^0 & \dots & A_{q-1}^0 & A_q^0 \\ I_n & 0 & \dots & 0 & 0 \\ 0 & I_n & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I_n & 0 \end{bmatrix},$$

$$\bar{A}_1 = \begin{bmatrix} A_0^1 & A_1^1 & \dots & A_{q-1}^1 & A_q^1 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \tag{21b}$$

$$\bar{A}_2 = \begin{bmatrix} A_0^2 & A_1^2 & \dots & A_{q-1}^2 & A_q^2 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

we can write (20) in the form

$$\bar{x}_{i+1,j+1} = \bar{A}_0 \bar{x}_{i,j} + \bar{A}_1 \bar{x}_{i+1,j} + \bar{A}_2 \bar{x}_{i,j+1}, \quad i,j \in \mathbb{Z}_+. \tag{22}$$

Therefore, the general model with q delays (20) has been reduced to the general 2D model without delays but with a higher dimension. Applying Theorem 1 to the model (22), we obtain the following result.

Theorem 7. *The general 2D model with q delays (20) is positive if and only if $A_k^t \in \mathbb{R}_+^{n\times n}$ for $t = 0,1,2$ and $k = 0,1,\dots,q$ or, equivalently, $\bar{A}_t \in \mathbb{R}_+^{\bar{N}\times\bar{N}}$ for $t = 0,1,2$.*

To check asymptotic stability of the positive 2D general model (22), we may use any of the conditions of Theorem 3.

In a similar way, the results can be extended to the positive general 2D model of the form

$$x_{i+1,j+1} = \sum_{k=0}^{q_1}\sum_{l=0}^{q_2} \big(A_{kl}^0 x_{i-k,j-l} + A_{kl}^1 x_{i-k+1,j-l} + A_{kl}^2 x_{i-k,j-l+1}\big), \quad i,j \in \mathbb{Z}_+, \tag{23}$$

where $x_{i,j} \in \mathbb{R}_+^n$ is the state vector at the point (i,j), and $A_{kl}^t \in \mathbb{R}_+^{n\times n}$, $t = 0,1,2$ and $k = 0,1,\dots,q_1$; $l = 0,1,\dots,q_2$.

4.2. 2D Roesser model with delays. Consider the autonomous positive 2D Roesser model with q delays in state

$$\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = \sum_{k=0}^{q} A_k \begin{bmatrix} x_{i-k,j}^h \\ x_{i,j-k}^v \end{bmatrix}, \quad i,j \in \mathbb{Z}_+, \tag{24}$$

where $x_{i,j}^h \in \mathbb{R}_+^{n_1}$ and $x_{i,j}^v \in \mathbb{R}_+^{n_2}$ are respectively the horizontal and vertical state vectors at the point (i,j), and

$$A_k = \begin{bmatrix} A_{11}^k & A_{12}^k \\ A_{21}^k & A_{22}^k \end{bmatrix}, \quad k = 0,1,\dots,q. \tag{25}$$

Defining the vectors

$$\bar{x}_{i,j}^h = \begin{bmatrix} x_{i,j}^h \\ x_{i-1,j}^h \\ \vdots \\ x_{i-q,j}^h \end{bmatrix}, \quad \bar{x}_{i,j}^v = \begin{bmatrix} x_{i,j}^v \\ x_{i,j-1}^v \\ \vdots \\ x_{i,j-q}^v \end{bmatrix}, \tag{26}$$

we can write (24) in the form

$$\begin{bmatrix} \bar{x}_{i+1,j}^h \\ \bar{x}_{i,j+1}^v \end{bmatrix} = A \begin{bmatrix} \bar{x}_{i,j}^h \\ \bar{x}_{i,j}^v \end{bmatrix}, \quad i,j \in \mathbb{Z}_+, \tag{27}$$

where A is defined in Eqn. (28).

Therefore, the 2D Roesser model with q delays (24) has been reduced to a 2D Roesser model without delays but with a higher dimension. Applying Theorem 2 to the model (27), we obtain the following result.

$$
A = \left[\begin{array}{ccccc|ccccc}
A_{11}^0 & A_{11}^1 & \cdots & A_{11}^{q-1} & A_{11}^q & A_{12}^0 & A_{12}^1 & \cdots & A_{12}^{q-1} & A_{12}^q \\
I_{n_1} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & I_{n_1} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
A_{21}^0 & A_{21}^1 & \cdots & A_{21}^{q-1} & A_{21}^q & A_{22}^0 & A_{22}^1 & \cdots & A_{22}^{q-1} & A_{22}^q \\
0 & 0 & \cdots & 0 & 0 & I_{n_2} & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & I_{n_2} & 0
\end{array}\right] \in \mathbb{R}^{N\times N}, \tag{28}
$$

$$N = (q+1)(n_1+n_2).$$

Theorem 8. *The 2D Roesser model with q delays (24) is positive if and only if $A_k \in \mathbb{R}_+^{(n_1+n_2)\times(n_1+n_2)}$ for $k = 0, 1, \ldots, q$ or, equivalently, $A \in \mathbb{R}_+^{N\times N}$.*

To check the asymptotic stability of the positive 2D Roesser model (24), we may use any of the conditions of Theorem 6.

In a similar way, the results can be extended for a positive 2D Roesser model of the form

$$
\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = \sum_{k=0}^{q_1}\sum_{l=0}^{q_2} A_{kl} \begin{bmatrix} x_{i-k,j-l}^h \\ x_{i-k,j-l}^v \end{bmatrix}, \quad i,j \in \mathbb{Z}_+, \tag{29}
$$

where $x_{i,j}^h \in \mathbb{R}^{n_1}$ and $x_{i,j}^v \in \mathbb{R}^{n_2}$ are respectively the horizontal and vertical state vectors at the point (i,j) and $A_{kl} \in \mathbb{R}^{(n_1+n_2)\times(n_1+n_2)}$.

5. Relationship between asymptotic stabilities of positive 2D linear systems with delays and positive 1D systems without delays

In this section it will be shown that the checking of asymptotic stability of positive 2D linear systems with delays can be reduced to testing asymptotic stability of the corresponding positive 1D linear systems without delays.

Theorem 9. *The positive system*

$$x_{i+1} = A_0 x_i + \sum_{k=1}^{q} A_k x_{i-k}, \tag{30}$$

where $A_k \in \mathbb{R}^{n\times n}$, $k = 0, 1, \ldots, q$ is asymptotically stable if and only if the positive system

$$x_{i+1} = A x_i, \quad A = \sum_{k=0}^{q} A_k \tag{31}$$

is asymptotically stable.

The proof is similar to that of Theorem 3 in (Busłowicz, 2008).

Theorem 10. *For the positive general 2D model (20), the following statements are equivalent:*

1. *The positive general 2D model is asymptotically stable.*
2. *All principal minors of the matrix*

$$I_n - \hat{A} = I_n - \sum_{k=0}^{q} A_k^0 + A_k^1 + A_k^2 \tag{32}$$

are positive.

3. *All coefficients of the characteristic polynomial of the matrix $\hat{A} - I_n = \left(\sum_{k=0}^{q} A_k^0 + A_k^1 + A_k^2\right) - I_n$,*

$$
\begin{aligned}
&\det\left[I_n(z+1) - \hat{A}\right] \\
&\quad = z^n + \hat{a}_{n-1}z^{n-1} + \cdots + \hat{a}_1 z + \hat{a}_0,
\end{aligned} \tag{33}
$$

are positive.

Proof. It is well known (Kaczorek, 2008a; 2008c) that the positive 2D general model without delays is asymptotically stable if and only if so is the corresponding positive 1D model. By Theorem 9, the positive 1D model with delays is asymptotically stable if and only if the corresponding positive 1D model without delays is asymptotically stable. Applying Theorem 3 to the corresponding positive 1D model, we obtain the equivalence of the statements (1), (2) and (3). ■

Remark 1. The positive general 2D model (20) is asymptotically stable if and only if the matrix $\hat{A}$ is a Schur matrix.

Theorem 11. *For the positive 2D Roesser model (24), the following statements are equivalent:*

1. *The positive 2D Roesser model is asymptotically stable.*
2. *All principal minors of the matrix*

$$I_n - \hat{A} = I_n - \sum_{k=0}^{q} \begin{bmatrix} A_{11}^k & A_{12}^k \\ A_{21}^k & A_{22}^k \end{bmatrix} \tag{34}$$

are positive.

3. *All coefficients of the characteristic polynomial of the matrix* $\hat{A}' - I_n$,

$$\det\left[I_n(z+1) - \hat{A}'\right] = z^n + \hat{a}'_{n-1}z^{n-1} + \cdots + \hat{a}'_1 z + \hat{a}'_0, \quad (35)$$

are positive.

The proof is similar to that of Theorem 10.

Remark 2. The positive 2D Roesser model (24) is asymptotically stable if and only if the matrix $\hat{A}$ is a Schur matrix.

Example 3. Using Theorem 10, check asymptotic stability of the positive 2D general model (20) for $q = 1$ with the matrices

$$A_0^0 = \begin{bmatrix} 0.1 & 0 \\ 0.2 & 0.2 \end{bmatrix}, \quad A_0^1 = \begin{bmatrix} 0.2 & 0 \\ 0.1 & 0.1 \end{bmatrix},$$
$$A_0^2 = \begin{bmatrix} 0.1 & 0 \\ 0.3 & 0.1 \end{bmatrix}, \quad A_1^0 = \begin{bmatrix} 0.2 & 0 \\ 0.4 & 0.1 \end{bmatrix}, \quad (36)$$
$$A_1^1 = \begin{bmatrix} 0 & 0 \\ 0.1 & 0.2 \end{bmatrix}, \quad A_1^2 = \begin{bmatrix} 0.1 & 0 \\ 0.3 & 0.1 \end{bmatrix}.$$

Using (32) and (33), we obtain

$$I_n - \hat{A} = \begin{bmatrix} 0.3 & 0 \\ -1.4 & 0.2 \end{bmatrix} \quad (37)$$

and

$$\det\left[I_n(z+1) - \hat{A}\right] = \begin{vmatrix} z+0.3 & 0 \\ -1.4 & z+0.2 \end{vmatrix} = z^2 + 0.5z + 0.06. \quad (38)$$

The principal minors of (37) are positive, i.e., $M_1 = 0.3$, $M_2 = 0.06$, and the coefficients of the polynomial (38) are positive. By Theorem 10, the positive general 2D model (20) with (36) is asymptotically stable.

6. Concluding remarks

Asymptotic stability of positive 2D linear systems with delays described by the general model and the Roesser model has been addressed. It was shown that

1. Asymptotic stability of positive 2D linear systems with delays is independent of the number and values of the delays and it depends only on the sum of the system matrices.
2. The checking of asymptotic stability of positive 2D linear systems with delays can be reduced to testing asymptotic stability of the corresponding positive 1D linear systems without delays.

The effectiveness of the proposed approach was demonstrated using numerical examples. The approaches can be also used for checking asymptotic stability of positive 2D linear systems with delays described by the first and second Fornasini-Marchesini models.

The extension of these results for positive 2D hybrid linear systems and 2D continuous-time systems is an open problem.

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