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ANTIPLANE DEFORMATION OF A BIMATERIAL CONTAINING AN INTERFACIAL CRACK WITH THE ACCOUNT OF FRICTION 2. REPEATING AND CYCLIC LOADING

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Abstract: The paper presents the exact solution of the antiplane problem for an inhomogeneous bimaterial with the interface crack exposed to the normal load and cyclic loading by a concentrated force in the longitudinal direction. Using discontinuity function method the problem is reduced to the solution of singular integral equations for the displacement and stress discontinuities at the domains with sliding friction. The paper provides the analysis of the effect of friction and loading parameters on the size of these zones. Hysteretic behaviour of the stress and displacement discontinuities in these domains is observed.

Keywords: Friction, Tribofatigue, Cyclic Loading, Interfacial Crack, Energy Dissipation, Stress Intensity Factor, Antiplane Deformation, Bimaterial, Discontinuity Function, Hysteresis

1. INTRODUCTION

The account of friction in studying of the contact phenomena is one of the most urgent problems of mechanical engineering and materials science in the analysis of the phenomena and processes occurring in moving elements of cars, during various technological operations (Goryacheva, 2001; Comninou, 1977; Panasiuk et al., 1976; Sulym and Piskozub, 2004; Johnson, 1985; Hills et al.,1993; Ulitko and Ostryk, 2006; Datsyshyn et al., 2006). Thus, friction can be accompanied with electric, thermal, vibrating and chemical processes, which damp the internal dynamic processes essentially influencing the intensity materials wear, and consequently the reliability and durability of the structural elements made of them (Sosnovskiy, 2005; Bogdanovich and Tkachuk, 2009; Evtushenko and Kutsei, 2010; Pasternak et al., 2010; Pyrievet al., 2012). Friction influence can be both negative and positive.

From the point of view of structural integrity mechanics, friction of crack faces at their relative displacement is useful in most cases, since it causes internal strain energy dissipation, and consequently reduces the stress concentration, which reduces or even eliminates alternating plastic deformations at alternating loading. It is also known that development of the residual stress field thus assists in the adaptation of a material to operational loadings. The compression of composite materials arising doe to friction forces improves shear stress redistribution even in the case of macroscopic fracture of a fiber-matrix interface.

Negative consequences of a friction are mainly the wear of contacting surfaces, and also thermal emission. At excessive intensity the latter can sometimes cause unpredictable change in mechanical, physical and chemical properties of a material, distribution of physical fields, and consequently influence the diffusive processes, in particular hydrogen diffusion, and development of the fracture phenomena warned by tribofatigue (Sosnovskiy, 2005; Evtushenko and Kutsei, 2010, Pyriev et al., 2012).

This paper continues previous authors' publication (Sulym et al., 2015) and develops the technique for studying the influence of friction in the antiplane problem for a solid with a closed crack under the applied quasi-static (inertia-free) repeatedly changing loading, including cyclic one. The most general case is considered, when at each step the loading can either increase (additional loading) or change sign (unloading) reaching sufficient magnitude, which causes development of slippage zones.

2. PROBLEM STATEMENT

Problem statement coincides with those resulted earlier in Sulym and Piskozub (2004) except the way of loading. Here it is supposed, that a medium is subject to repeatedly changing loading, which cause the quasi-static antiplane deformation of a solid and corresponding stress strain state (in-plane loading is assumed to be constant). As the special case the cyclically changing loading can be considered, which is performed within the pattern loading-unloading-loading-...

As well as in the previous work (Sulym et al., 2015), consider an infinite isotropic medium consisting of two half-spaces with elastic constants $E_k,\ \nu_k,\ G_k \quad (k=1,2)$, which are pressed to each other along their interface L with normal stress $\sigma_{yyk} = -P \ \ (k=1,2;\ x \in L)$. Here the system of co-ordinates $Oxyz$ is used, with its origin at a plane xOz of contact of half-spaces, where NOz-coaxial strip cracks are localized at $L' = \bigcup_{n=1}^{N} L'_n = \bigcup_{n=1}^{N} [a_n^-;\ a_n^+]$ (Fig. 1).

Thus, the problem is reduced to study of stress strain state (SSS) of a cross-section xOy perpendicular to a direction z of its longitudinal (out-of-plane) displacement. The half-spaces perpendicular to this axis form to half-planes $S_k \ \ (k=1,2)$, and their interface correspond to the abscissa $L{\sim}x$.

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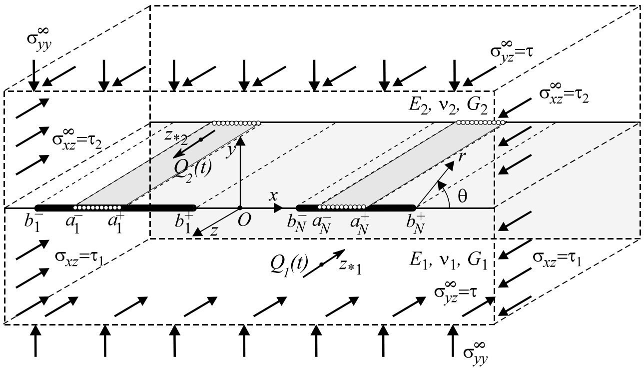

Fig. 1. The loading and geometric scheme of the problem

Thus, the problem is reduced to study of stress strain state (SSS) of a cross-section xOy perpendicular to a direction z of its longitudinal (out-of-plane) displacement. The half-spaces perpendicular to this axis form to half-planes S_k $(k = 1,2)$, and their interface correspond to the abscissa $L \sim x$.

The application of similar traditional notation for an axis z and a complex variable $z = x + iy$ should not cause misunderstanding in the solution of the problem.

Contact between the bimaterial medium components along a line $L'' = L \backslash L'$ is supposed to be mechanically perfect, and the contact along defects' (cracks') faces L' is assumed to be performed according to the laws of tangential mechanic contact, at which bodies contact mechanically perfect until the moment, when relative sliding of crack faces may start in some areas $\gamma_n^{(p)} \subset L'_n$ at the material interface (Johnson, 1985; Sulym et al., 2015).

Presence of such slippage zones (cracks with contacting faces) at each p-th step of loading (cycle) is modeled with stress and displacement discontinuity vectors at $\gamma_n^{(p)} \subset L'_n$ (Bozhydarnyk and Sulym, 1999; Sulym, 2007; Piskozub and Sulim, 2008):

$$[\Xi]_{L'} \equiv \Xi^- - \Xi^+ = \mathbf{f}_{(p)}(x,t), \tag{1}$$

for $x \in \gamma_n^{(p)} \subset L'_n$ $(n = \overline{1,N})$,

$$\mathbf{f}_{(p)}(x,t) = 0, \text{ if } x \notin \gamma_n^{(p)} \subset L'_n, \tag{2}$$

where $\Xi(z,t) = \{\sigma_{yz}, \partial w/\partial x\}(z,t)$ is a state vector; $\mathbf{f}_{(p)}(x,t) = \{f_{3(p)}, f_{6(p)}\}(x,t)$ is a discontinuity vector; p in brackets denote the number of loading step (cycle); t is time as a formal monotonously increasing parameter related with the convertible loading. The following notation are used hereinafter: $[\phi] = \phi(x,-0) - \phi(x,+0)$, $\langle\phi\rangle = \phi(x,-0) + \phi(x,+0)$; indices "+" and "–" correspond to the limit values of a function at the top and bottom edges of a line L.

The friction contact conditions at the closed crack provide that at the achievement by tangent traction σ_{yz} at the lines $\gamma_n^{(p)}$ of a certain critical value $\tau_{yz}^{\max}$ the slippage occurs, and the tangent traction cannot exceed this threshold. Thus, within the classical Amontons' law of friction (Johnson, 1985), consider a variant of a contact problem according to which the tangent traction (friction traction) is constant along the lines $\gamma_n^{(p)}$:

$$\begin{aligned} \sigma_{yz}^{\pm} &= -sgn([w]_{(p)})\tau_{yz}^{max}, \\ \tau_{yz}^{max} &= -\alpha\sigma_{yy}(|w^- - w^+| \neq 0)' \end{aligned} \tag{3}$$

where α is a coefficient of dry friction. Outside the lines $\gamma_n^{(p)}$, which belong to L'_n, the tangent traction at the crack points without slippage does not exceed the possible admissible level

$$|\sigma_{yz}| \le \tau_{yz}^{\max}(w^- - w^+ = 0), \tag{4}$$

and the mutual crack face displacement (displacement discontinuity) is absent. The sign (an action direction) of tangent traction is chosen depending on a sign of the difference of displacements $[w]_{(p)}$ at a considered point of $\gamma_n^{(p)}$.

3. THE PROBLEM SOLUTION

Assume that the magnitude and direction of action of the external mechanical loading factors, which perform the longitudinal shear of a medium, change quasi-statically (so slowly that there is no necessity to consider inertial terms) under the certain law which can be arbitrary. Let the external loading of the problem be defined by monotonously changing in time intervals $[t_{(p-1)};\ t_{(p)}]$ step-by-step sequences of the following factors: stress $\sigma_{yz}^{\infty} = \sum_p \tau_{(p)}(t)$, $\sigma_{xz}^{\infty} = \sum_p \tau_{k(p)}(t)$ uniformly distributed at the infinity; the concentrated forces with magnitude $Q_k(t) = \sum_p Q_{k(p)}(t)$, and screw dislocations with Burgers vectors $b_k(t) = \sum_p b_{k(p)}(t)$ applied at the points $z_{*k} \in S_k (k = 1, 2)$. It should be noticed that the positive direction of force and Burgers vectors is selected along z-axis (such that it along with x and y axes forms the right rectangular coordinate system), unlike Panasyuk et al. (1976), where the opposite directions is implicitly accepted as a positive one. According to (20.5) Sulym (2007), at each moment of time stress at the infinity should satisfy the condition:

$$\tau_{2(p)}(t)G_1 = \tau_{1(p)}(t)G_2, \tag{5}$$

which provides straightness of the material interface at the infinity.

The first (initial) step of loading and the SSS produced by it are considered in details in Ref [17], where the resulting system of singular integral equations (SSIE) is obtained

$$\begin{cases} f_{3(1)}(x,t) = 0, & (x \in L'), \\ g_{6(1)}(x,t) = \frac{1}{2C}\left(\langle\sigma_{yz(1)}^0(x,t)\rangle + 2\text{sgn}[w]_{(1)}\tau_{yz}^{\max}\right), \end{cases} \tag{6}$$

which has the following closed-form solution:

$$\begin{aligned} &f_{6(1)}(x,t) = \frac{X_0^{*+}(x)}{\pi i}\int_{L'}\frac{F_{6(1)}(s,t)ds}{X_0^{*+}(s)(s-x)} + \\ &X_0^{*+}(x)Q_{n-1}(x), (x \in L'), \\ &X_0^{*}(z) = \prod_{n=1}^{N}\left[\left(z - a_{n(1)}^-\right)\left(z - a_{n(1)}^+\right)\right]^{-1/2}, \end{aligned} \tag{7}$$

where the factors at polynomials $Q_{n-1}(x)$ are determined from additional displacement continuity conditions at each crack:

$$\int_{a_{n(1)}^-}^{a_{n(1)}^+} f_{6(1)}(s,t)ds = 0 \quad \left(n = \overline{1,N}\right). \tag{8}$$

Here and further for each step the following notation is used (Sulym et al., 2015):

$$\begin{aligned} &\sigma_{yz(p)}^0(z,t) + i\sigma_{xz(p)}^0(z,t) = \tau_{(p)}(t) + i\{\tau_{k(p)}(t) \\ &+D_{k(p)}(z,t) + (p_k - p_j)\overline{D}_{k(p)}(z,t) + 2p_k D_{j(p)}(z,t)\}, \\ &D_{k(p)}(z,t) = -\frac{Q_{k(p)}(t) + iG_k b_{k(p)}(t)}{2\pi(z - z_{*k})}, (z \in S_k, \\ &\qquad k = 1, 2), \\ &g_{r(p)}(z,t) = \frac{1}{\pi}\int_{L'}\frac{f_{r(p)}(x,t)dx}{x - z}, \\ &C = \frac{G_1 G_2}{G_1 + G_2}, \qquad p_k = \frac{C}{C_j}. \end{aligned} \tag{9}$$

and SSS components are defined by relations

$$\begin{aligned}&\sigma_{yz(1)}(z,t)+i\sigma_{xz(1)}(z,t)=\sigma^0_{yz(1)}(z,t)\\&+i\sigma^0_{xz(1)}(z,t)+ip_k g_{3(1)}(z,t)-\ Cg_{6(1)}(z,t)\\&(z\in S_k;\ \ r=3,6;\ \ k=1,2;\ \ j=3-k);\\&\sigma^{\pm}_{yz(1)}(x,t)=\mp p_k f_{3(1)}(x,t)\\&-Cg_{6(1)}(x,t)+\sigma^{0\pm}_{yz(1)}(x,t),\\&\sigma^{\pm}_{xz(1)}(x,t)=\mp Cf_{6(1)}(x,t)+p_k g_{3(1)}(x,t)\\&+\sigma^{0\pm}_{xz(1)}(x,t),\qquad (x\in L').\end{aligned}\tag{10}$$

Consider the next step of loading. Assume that the components of SSS of the medium obtained at the end time $t_{(1)}$ of the previous (first) step can be considered as residual ones.

Then one can assume that the problem statement at this step differs from the formulation of the problem of the previous step in already existing displacement and stress discontinuities caused by the previous step of loading. Hence, at this step additional change in loading is accompanied with additional discontinuities and the representation of a total stress field, which account for the residual SSS from the previous ($p=1$) step, is as follows:

$$\begin{aligned}&\sigma_{yz}(z,t)+i\sigma_{xz}(z,t)=\sigma_{yz(1)}\big(z,t_{(1)}\big)\\&+i\sigma_{xz(1)}\big(z,t_{(1)}\big)+\sigma^0_{yz(2)}(z,t)+i\sigma^0_{xz(2)}(z,t)\\&+ip_k g_{3(2)}(z,t)-Cg_{6(2)}(z,t),\\&(z\in S_k;\ \ k=1,2;\ \ j=3-k).\end{aligned}\tag{11}$$

The total stress should satisfy the boundary conditions (3) at $\gamma_n^{(2)}$ with the account of a loading direction. Then one can formulate the following local problem for the second step:

$$\begin{aligned}&\sigma_{yz(2)}(z,t)+i\sigma_{xz(2)}(z,t)=\{\sigma_{yz}(z,t)+i\sigma_{xz}(z,t)\}\\&-\{\sigma_{yz(1)}\big(z,t_{(1)}\big)+i\sigma_{xz(1)}\big(z,t_{(1)}\big)\}\\&(z\in S_k;\ \ k=1,2;\ \ j=3-k)\end{aligned}\tag{12}$$

with boundary conditions

$$\begin{aligned}&\sigma^{\pm}_{yz(2)}(x,t)=-\mathrm{sgn}([w]_{(2)})\tau^{\max}_{yz}-\sigma^{\pm}_{yz(1)}\big(x,t_{(1)}\big),\\&x\in\gamma_n^{(2)}\subset L'_n\ (n=\overline{1,N}).\end{aligned}\tag{13}$$

Conditions (13) can be specified depending on a relation between $\gamma_n^{(2)}$ and $\gamma_n^{(1)}$:

$$\begin{aligned}&\sigma^{\pm}_{yz(2)}(x,t)\\&=\begin{cases}-\mathrm{sgn}([w]_{(2)})\tau^{\max}_{yz}+\mathrm{sgn}([w]_{(1)})\tau^{\max}_{yz}, & x\in\gamma_n^{(2)}\subset\gamma_n^{(1)}\\-\mathrm{sgn}([w]_{(2)})\tau^{\max}_{yz}-\sigma^{\pm}_{yz(1)}\big(x,t_{(1)}\big), & x\in\gamma_n^{(2)}\backslash\gamma_n^{(1)}.\end{cases}\end{aligned}\tag{14}$$

According to Eq (14), the abovementioned assumption does not demand the proof in a case, when at a following step the sign of applied (local at this step) loading changes. As soon as at the moment $t_{(1)}$ of the first step end the local loadings reach their extreme values (at their increase those are maxima), a slippage zone is fixed in size, and contact surfaces stick together and the reached SSS is further considered as residual one. After that, with the beginning of a following step the magnitudes of total loadings start to decrease and quite similar to the process of unloading of the plastic material, new slippage does not arise, while at certain time $t^{st}_{(2)}\big(t_{(1)}<t^{st}_{(2)}\le t_{(2)}\big)$ the slippage conditions (3) are to be satisfied. Thus, the starting size of a slippage zone at the second step is always less than its size in the end of the previous step: $\gamma_n^{(2)}(t^{st}_{(2)})\subset\gamma_n^{(1)}(t_{(1)})$. Therefore, using for this case of loading the reasoning similar to that of the previous step one can obtain the following resulting SSIE

$$\begin{cases}f_{3(2)}(x,t)=0,\\ g_{6(2)}(x,t)=\dfrac{1}{2C}\{\langle\sigma^0_{yz(2)}(x,t)\rangle+2\tau^{\max}_{yz}\big(\mathrm{sgn}[w]_{(2)}-\mathrm{sgn}[w]_{(1)}\big)\};\end{cases}$$

for determination of local (additional regarding the SSS reached at time $t_{(1)}$) stress and displacement discontinuities from local (for this step) loadings. The obtained solution differs from Eqs (7)–(10) only in the influence of additional term $-2\tau^{\max}_{yz}\mathrm{sgn}[w]_{(1)}$ at the right hand side of SSIE (15), and of course, the superscript in brackets defines the second step.

Providing similar reasoning for the following steps of alternating monotonously changing loading one can obtain the local problem for the p-th step:

$$\begin{aligned}&\sigma_{yz(p)}(z,t)+i\sigma_{xz(p)}(z,t)=\{\sigma_{yz}(z,t)+i\sigma_{xz}(z,t)\}\\&-\sum_{i=1}^{p-1}\{\sigma_{yz(i)}\big(z,t_{(i)}\big)+i\sigma_{xz(i)}\big(z,t_{(i)}\big)\}\\&\big(z\in S_k;\ \ k=1,2;\ \ j=3-k;\ \ t>t_{(p-1)}\big)\end{aligned}\tag{16}$$

with boundary conditions:

$$\begin{aligned}&\sigma^{\pm}_{yz(p)}(x,t)=-\mathrm{sgn}([w]_{(p)})\tau^{\max}_{yz}-\sigma^{\pm}_{yz(p-1)}\big(x,t_{(m)}\big)\\&=-\tau^{\max}_{yz}\big(\mathrm{sgn}[w]_{(p)}-\mathrm{sgn}[w]_{(p-1)}\big),\\&x\in\gamma_n^{(p)}\subset L'n\ (n=\overline{1,N}),\end{aligned}\tag{17}$$

which results in the following SSIE

$$\begin{cases}f_{3(p)}(x,t)=0,\\ g_{6(p)}(x,t)=\dfrac{1}{2C}\{\langle\sigma^0_{yz(p)}(x,t)\rangle+2\tau^{\max}_{yz}\big(\mathrm{sgn}[w]_{(p)}-\mathrm{sgn}[w]_{(p-1)}\big)\};\end{cases}\tag{18}$$

and its solution has the same structure as in Eqs (7)–(10).

In general the local displacement discontinuity and energy dissipation at the p-th step are defined as:

$$\begin{aligned}&[w]_{(p)}(x,t)=\int_{a^-_{n(p)}}^{x}f_{6(p)}(s,t)ds,\\&x\in\gamma_n^{(p)}\subset L'\mathrm{n},\ (n=\overline{1,N}),\end{aligned}\tag{19}$$

$$W^d_{(p)}(t)=-\int_{L'}\tau^{\max}_{yz}\big|[w]_{(p)}(x,t)\big|\,dx.\tag{20}$$

As a consequence, total values of stress, strain, displacement and its discontinuity, the dissipated energy etc. after the p-th step can be presented as a superposition, for instance

$$\begin{aligned}&[w](x,t)=\textstyle\sum_{m=1}^{p-1}[w]_{(m)}(x,t_{(m)})\ +[w]_{(p)}(x,t)\ ,\\&\big(x\in L';\ \ t>t_{(p-1)}\big)\end{aligned}\tag{21}$$

$$W^d(t)=\textstyle\sum_{m=1}^{p-1}W^d_{(m)}\big(t_{(m)}\big)\ +W^d_{(p)}(t),\ \big(t>t_{(p-1)}\big).\tag{22}$$

As well as in Sulym et al. (2015), for detailed illustration of the developed approach for solution of the problem consider a special case of alternating loading symmetric concerning a vertical axis ($z_{*k}=\pm id$) of a medium containing a single ($N=1$) crack $L'_1=[-b;\ b]$. Then in the bounds of L'_1 at each loading step only a symmetric slippage zone $\gamma_1^{(p)}=[-a_{(p)};\ a_{(p)}]\big(a_{(p)}\le b\big)$ can occur and

$$\langle\sigma^0_{yz(p)}(x,t)\rangle=2\tau_{(p)}(t)-4p_1\mathrm{Im}D_{2(p)}(x,t)-4p_2\mathrm{Im}D_{1(p)}(x,t),\qquad Q_0(x)\equiv c_0=0.$$

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In this case the solution of the integral equation (18) after calculation of corresponding integrals is as follows:

$$f_{6(p)}(x,t)=\frac{1}{\pi C\sqrt{a_{(p)}^2-x^2}}\{\pi\left(\tau_{(p)}(t)+\tau_{yz}^{\max}\left(\operatorname{sgn}[w]_{(p)}-\operatorname{sgn}[w]_{(p-1)}\right)\right)x+$$
$$+\sum_{k=1}^{2}p_{2-k}\left(Q_{k(p)}(t)\operatorname{Im}\frac{\sqrt{{z_{*k}}^2-a_{(p)}^2}}{x-z_{*k}}+G_k b_{k(p)}(t)\operatorname{Re}\left(\frac{\sqrt{{z_{*k}}^2-a_{(p)}^2}}{x-z_{*k}}+1\right)\right)\Bigg\},\qquad \left(x\in[-a_{(p)};\ a_{(p)}]\right). \tag{23}$$

The function $X(z)=\sqrt{z^2-a^2}$ is understand as a branch, satisfying the condition $\sqrt{z^2-a^2}/z\to 1$ as $z\to\infty$. Similar reasoning is used for a choice of branches of functions $\sqrt{z_{*k}^2-a^2}$and $\sqrt{\bar{z}_{*k}^2-a^2}$, $k=1,\ 2.$

Based on Eq (23) one can obtain the following formula for $g_{6(p)}(z,t)$:

$$g_{6(p)}(z,t)=\frac{1}{C}\left[\tau_{(p)}(t)+\tau_{yz}^{\max}\left(\operatorname{sgn}[w]_{(p)}-(\operatorname{sgn}[w]_{(p-1)}\right)\right]\left(1-\frac{z}{\sqrt{z^2-a_{(p)}^2}}\right)-\frac{p_1G_2b_{2(p)}(t)+p_2G_1b_{1(p)}(t)}{\pi C\sqrt{z^2-a_{(p)}^2}}$$
$$-\frac{1}{2\pi C}\sum_{k=1}^{2}p_{2-k}\left(iQ_{k(p)}(t)R_k^-\left(a_{(p)},z,z_{*k}\right)-\left(G_k b_{k(p)}(t)R_k^+\left(a_{(p)},z,z_{*k}\right)\right)\right),\quad \left(z\notin[-a_{(p)};\ a_{(p)}]\right), \tag{24}$$

where:

$$R_k^{\pm}(a,z,z_{*k})=\frac{1}{\pi}\int_{-a}^{a}\left(\frac{\sqrt{z_{*k}^2-a^2}}{x-z_{*k}}\pm\frac{\sqrt{\bar{z}_{*k}^2-a^2}}{x-\bar{z}_{*k}}\right)\frac{dx}{\sqrt{a^2-x^2}(x-z)}=\frac{1}{\sqrt{z^2-a^2}}\left(\frac{\sqrt{z_{*k}^2-a^2}}{z_{*k}-z}\pm\frac{\sqrt{\bar{z}_{*k}^2-a^2}}{\bar{z}_{*k}-z}\right)-\left(\frac{1}{z_{*k}-z}\pm\frac{1}{\bar{z}_{*k}-z}\right).$$

Expression for displacement discontinuity $[w]_{(p)}$ can be obtained from Eqs (19), (23) as

$$[w]_{(p)}(x,t)=\int_{-a_{(p)}}^{x}f_{6(p)}(s,t)ds=-\frac{1}{C}\left(\tau_{(p)}(t)+\tau_{yz}^{\max}\left(\operatorname{sgn}[w]_{(p)}-\operatorname{sgn}[w]_{(p-1)}\right)\right)\sqrt{a^2{}_{(p)}-x^2}+$$
$$+\frac{1}{\pi C}\{\sum_{k=1}^{2}p_{2-k}\left(Q_{k(p)}(t)\operatorname{Im}I(x,z_{*k})+G_k b_{k(p)}(t)\left(\pi+2\arcsin\frac{x}{a_{(p)}}+\operatorname{Re}I(x,a_{(p)},z_{*k})\right),\quad \left(|x|\le a_{(p)}\right),\right. \tag{25}$$

where: $I(x,a,z)\equiv\sqrt{z^2-a^2}\int_{-a}^{x}\frac{dx}{\sqrt{a^2-t^2}(x-z)}=i\ln\frac{a(z-x)}{a^2-xz-i\sqrt{a^2-x^2}\sqrt{z^2-a^2}}$.

From Eqs (20), (25) it follows the expression for energy dissipation $W_{(p)}^{d}(t)$:

$$W_{(p)}^{d}(t)=-\int_{-a_{(p)}}^{a_{(p)}}\tau_{yz}^{\max}\left|[w]_{(p)}(x,t)\right|dx=-\frac{\tau_{yz}^{\max}}{C}\Bigg|\frac{\pi a_{(p)}^2}{2}\left(\tau_{(p)}(t)+\tau_{yz}^{\max}\left(\operatorname{sgn}[w]_{(p)}-\right.\right.$$
$$\left.\left.-\operatorname{sgn}[w]_{(p-1)}\right)\right)+\sum_{k=1}^{2}p_{2-k}\left(Q_{k(p)}(t)\right)\operatorname{Im}\left(\sqrt{z_{*k}^2-a_{(p)}^2}-z_{*k}\right)+G_k b_{k(p)}(t)\operatorname{Re}\left(\sqrt{z_{*k}^2-a_{(p)}^2}-z_{*k}\right)\Bigg|. \tag{26}$$

Consider in details the determination of the size $a_{(p)}$ of the slippage zone at each step of loading. Here SIF is the defining parameter, which is determined within Eq (22) of Sulym and Piskozub (2004) as:

$$K_{3(p)}(t)=\frac{1}{\sqrt{\pi a_{(p)}}}\int_{-a_{(p)}}^{a_{(p)}}\sqrt{\frac{a_{(p)}\pm x}{a_{(p)}\mp x}}\sigma_{yz}(x,t)\,dx==\frac{1}{\sqrt{\pi a_{(p)}}}\int_{-a_{(p)}}^{a_{(p)}}\sqrt{\frac{a_{(p)}\pm x}{a_{(p)}\mp x}}\left\{\sum_{m=1}^{p-1}\sigma_{yz(m)}\left(x,t_{(m)}\right)+\sigma_{yz(p)}^{0}(x,t)+\operatorname{sgn}[w]_{(p)}\tau_{yz}^{\max}\right\}dx$$
$$=\sqrt{\pi a_{(p)}}\left(\sum_{m=1}^{p-1}\tau_{(m)}(t_{(m)})+\tau_{(p)}(t)+\operatorname{sgn}[w]_{(p)}\tau_{yz}^{\max}\right)-\frac{1}{\sqrt{\pi a_{(p)}}}\sum_{k=1}^{2}p_{2-k}\left\{\left(\sum_{m=1}^{p-1}Q_{k(m)}(t_{(m)})+Q_{k(p)}(t)\right)\operatorname{Im}\frac{a_{(p)}\pm z_{*k}}{\sqrt{z_{*k}^2-a_{(p)}^2}}+\right.$$
$$\left.+\left(\sum_{m=1}^{p-1}b_{k(m)}(t_{(m)})+b_{k(p)}(t)\right)G_k\operatorname{Re}\left(\frac{a_{(p)}\pm z_{*k}}{\sqrt{z_{*k}^2-a_{(p)}^2}}\mp 1\right)\right\}. \tag{27}$$

The equality of SIF to zero provides the condition for slippage start at the p-th step, the magnitude of the first critical loading $Q_{k(p)}^{*}$, and the size $a_{(p)}$ of the slippage.

For definiteness it is assumed (other cases are studied similarly) that at the point $z_{*2}=id$ of the top half-space only one alternating monotonously changing concentrated force with magnitude $Q_2(t)=\sum_m Q_{2(m)}(t)$ is applied, which increase at odd and decrease at even m. SIF magnitude, the size of a slippage zone, displacement discontinuity and energy dissipation at the first step of such loading are studied in Ref [17]. Consider the second step (unloading), when $Q_{2(2)}(t)<0$ $\left(t>t_{(1)}\right)$. Accounting for the fact that $\operatorname{sgn}[w]_{(2)}=1$ at this step, from expression (27) one can obtain the size of new slippage zone

$$a_{(2)}(t)=\sqrt{\frac{p_1^2Q_{2(2)}(t)^2}{4\pi^2{\tau_{yz}^{\max}}^2}-d^2}, \tag{28}$$

and a condition at unloading, when the slippage starts over again

$$\left|Q_{2(2)}(t)\right|\ge\frac{2\pi d\tau_{yz}^{\max}}{p_1}=Q_{2(2)}^{*}=2Q_{2(1)}^{*}. \tag{29}$$

Here the first critical value of the applied force at the p-th step is denoted as $Q_{2(p)}^{*}$. Local displacement discontinuity and the energy dissipation at the second step for such loading (while the condition (29) holds) is as follows

$$[w]_{(2)}(x,t)=\int_{-a_{(2)}}^{x} f_{6(2)}(x,t)dx=\frac{p_1 Q_{2(2)}(t)}{2\pi C}\ln\frac{\sqrt{a_{(2)}^2+d^2}-\sqrt{a_{(2)}^2-x^2}}{\sqrt{a_{(2)}^2+d^2}+\sqrt{a_{(2)}^2-x^2}}-\frac{2}{C}\tau_{yz}^{\max}\sqrt{a_{(2)}^2-x^2}\left(|x|\le a_{(2)}\right); \tag{30}$$

$$W_{(2)}^d(t)=-\int_{-a_{(2)}}^{a_{(2)}}\tau_{yz}^{\max}\left|[w]_{(2)}(x,t)\right|dx=\frac{\pi a_{(2)}^2 \tau_{yz}^{\max 2}}{2C}-\frac{\tau_{yz}^{\max}}{C}p_1 Q_{2(2)}(t)\left(\sqrt{a_{(2)}^2+d^2}-d\right). \tag{31}$$

Assuming that $a=b$ in (29) one can obtain the second critical force, at which nonzero SIF (singular stress) arise at the vicinity of crack tips

$$Q_{2(2)}^{**}=\frac{2\pi\tau_{yz}^{\max}}{p_1}\sqrt{d^2+b^2}=2Q_{2(2)}^{*}\frac{\sqrt{d^2+b^2}}{d}.$$

Continuing similar reasoning for the following steps of loadings one can find that both critical loadings of the second and following steps is higher twice than corresponding critical loadings of the initial step.

For smooth contact between crack faces (zero friction coefficient) one should assume $\tau_{yz}^{\max}=0$ in the abovementioned equations. This special case coincides with the solution of the antiplane problem for an interfacial crack under the identical static loading in Panasyuk et al. (1976) and in Sulym (2007).

Let's prove that the proposed additive approach of the account of repeating loading is suitable and for the case when at the following step the applied loading does not change its sign. As at time $t_{(1)}$ the first step loading reaches maximum, and then continues to increase already at the second step, the slippage zon after the first step continues to grow without a delay to a maximum $\gamma_n^{(2)}(t)\supset\gamma_n^{(1)}(t_{(1)})$ $\left(t_{(2)}^{st}=t_{(1)}\right)$ of the second. Hence, if the total solution after the second step obtained by means of Eqs (23)–(28) coincides with the solution of this problem in a single step, but the medium is then subjected to the total loading of two steps. Thus, the proposed technique can be used for the arbitrary quasi-static multistage loading. Let's show this on the abovementioned example for the two first steps of action of symmetric loading with the regard to the vertical axis ($z_{*k}=\pm id$) of a medium with a single ($N=1$) crack. For simplification of calculations we will assume that only one concentrated force $Q_{2(p)}(t)$ acts in the medium. Since $a_{(2)}(t)\ge a_{(1)}(t_{(1)})$ for $t\ge t_{(1)}$, in this case one should account for the second part of the boundary conditions (14), which contains the following terms:

$$\sigma_{yz(1)}^{\pm}(x,t_{(1)})=\sigma_{yz(1)}^{0\pm}(x,t_{(1)})-Cg_{6(1)}(x,t_{(1)})=\sigma_{yz(1)}^{0\pm}(x,t_{(1)})-\frac{C}{\pi}\int_{-a_{(1)}}^{a_{(1)}}\frac{f_{6(1)}(\xi,t_{(1)})d\xi}{\xi-x},$$
$$x\subset[-a_{(2)};a_{(2)}]\backslash[-a_{(1)};a_{(1)}]. \tag{32}$$

Calculating $K_{3(2)}(t)$ at the second step one obtains

$$\begin{aligned}
K_{3(2)}(t)&=\frac{1}{\sqrt{\pi a_{(2)}}}\int_{-a_{(2)}}^{a_{(2)}}\sqrt{\frac{a_{(2)}\pm x}{a_{(2)}\mp x}}\left\{\sigma_{yz(1)}(x,t_{(1)})+\sigma_{yz(2)}^0(x,t)+\operatorname{sgn}[w]_{(2)}\tau_{yz}^{\max}\right\}dx=\\
&=\frac{1}{\sqrt{\pi a_{(2)}}}\int_{-a_{(2)}}^{a_{(2)}}\sqrt{\frac{a_{(2)}\pm x}{a_{(2)}\mp x}}\left(\sigma_{yz(2)}^0(x,t)+\operatorname{sgn}[w]_{(2)}\tau_{yz}^{\max}+\begin{cases}-\operatorname{sgn}[w]_{(1)}\tau_{yz}^{\max}, & x\in\gamma_1^{(2)}\subset\gamma_1^{(1)}\\ \sigma_{yz(1)}^0(x,t)-Cg_{6(1)}(x,t_{(1)}), & x\in\gamma_1^{(2)}\backslash\gamma_1^{(1)}\end{cases}\right)dx=\\
&=\frac{1}{\sqrt{\pi a_{(2)}}}\int_{-a_{(2)}}^{a_{(2)}}\sqrt{\frac{a_{(2)}\pm x}{a_{(2)}\mp x}}\left(\sigma_{yz(2)}^0(x,t)+\operatorname{sgn}[w]_{(2)}\tau_{yz}^{\max}\right)dx-\frac{\operatorname{sgn}[w]_{(1)}\tau_{yz}^{\max}}{\sqrt{\pi a_{(2)}}}\int_{-a_{(1)}}^{a_{(1)}}\sqrt{\frac{a_{(2)}\pm x}{a_{(2)}\mp x}}dx+\\
&+\frac{1}{\sqrt{\pi a_{(2)}}}\int_{-a_{(2)}}^{-a_{(1)}}\sqrt{\frac{a_{(2)}\pm x}{a_{(2)}\mp x}}\left(\sigma_{yz(1)}^0(x,t)-Cg_{6(1)}(x,t_{(1)})\right)dx+\\
&+\frac{1}{\sqrt{\pi a_{(2)}}}\int_{a_{(1)}}^{a_{(2)}}\sqrt{\frac{a_{(2)}\pm x}{a_{(2)}\mp x}}\left(\sigma_{yz(1)}^0(x,t)-Cg_{6(1)}(x,t_{(1)})\right)dx.
\end{aligned} \tag{33}$$

Accounting for $\operatorname{sgn}[w]_{(2)}=\operatorname{sgn}[w]_{(1)}=-1$ in the considered case, and utilizing the values of integrals

$$\begin{aligned}
&\frac{1}{\pi}\int_{-a}^{a}\frac{\xi d\xi}{\sqrt{a^2-\xi^2}(\xi-x)}=1-\frac{|x|}{\sqrt{x^2-a^2}},\\
&\frac{1}{\pi}\int_{-a}^{a}\frac{d\xi}{\sqrt{a^2-\xi^2}(\xi-x)(\xi-z)}=\frac{1}{z-x}\left(-\frac{1}{\sqrt{z^2-a^2}}+\frac{\operatorname{sgn}(x)}{\sqrt{x^2-a^2}}\right)x\notin[-a;a],\\
&\int_{-b}^{-a}\sqrt{\frac{b\pm\xi}{b\mp\xi}}d\xi+\int_{a}^{b}\sqrt{\frac{b\pm\xi}{b\mp\xi}}d\xi=2b\left(\frac{\pi}{2}-\arcsin\frac{a}{b}\right),\qquad \int_{-b}^{-a}\sqrt{\frac{b\pm\xi}{b\mp\xi}}\frac{|\xi|d\xi}{\sqrt{\xi^2-a^2}}+\int_{a}^{b}\sqrt{\frac{b\pm\xi}{b\mp\xi}}\frac{|\xi|d\xi}{\sqrt{\xi^2-a^2}}=b\pi,\\
&\int_{-b}^{-a}\sqrt{\frac{b\pm\xi}{b\mp\xi}}\frac{\operatorname{sgn}(\xi)d\xi}{(\xi-id)\sqrt{\xi^2-a^2}}+\int_{a}^{b}\sqrt{\frac{b\pm\xi}{b\mp\xi}}\frac{\operatorname{sgn}(\xi)d\xi}{(\xi-id)\sqrt{\xi^2-a^2}}=\frac{\pi(b\pm id)}{\sqrt{a^2+d^2}\sqrt{b^2+d^2}},
\end{aligned} \tag{34}$$

one obtains SIF as:

$$K_{3(2)}(t)=\sqrt{\pi a_{(2)}}\left(\tau_{(1)}(t_{(1)})+\tau_{(2)}(t)-\tau_{yz}^{\max}\right)-\sqrt{\frac{a_{(2)}}{\pi}}\frac{p_1\left(Q_{2(1)}(t_{(1)})+Q_{2(2)}(t)\right)}{\sqrt{a_{(2)}^2+d^2}}, \tag{35}$$

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which coincides with the sum of expressions (25) and (26) of Sulym et al. (2015) for the case of a single-step loading equal to $\tau_{(1)}(t_{(1)}) + \tau_{(2)}(t)$ and $Q_{2(1)}(t_{(1)}) + Q_{2(2)}(t)$.

Thus, the proposed additive approach to the sequence of residual SSS is suitable for the account of multistage loading-unloading. However, for simplification of the solution procedure it is expediently to unite the consecutive steps of additional loading or unloading in one single step. In this case, the solution of the problem formulated in section 1 can be easily obtained by means of the above-stated technique for alternating loading.

4. THE NUMERICAL ANALYSIS

Consider the following dimensionless values, which significantly reduce the amount of calculations without loss in generality: $a_{(p)}/b$, x/b, d/b are the slippage zone, co-ordinate x and distance to the concentrated force application point, respectively, normalized to the semi-length of a crack at the p-th step; $\beta_{(p)} = Q_{(p)}(t)/Q^*_{(1)}$ is a normalized magnitude of the applied force at the p-th step; $\pi dC[w]_{(p)}(x,t)/bp_1Q^*_{(1)}$, $CW_{(p)}(x,t)/\pi\alpha^2b^2P^2$ are normalized displacement discontinuity and the energy dissipation at the p-th step.

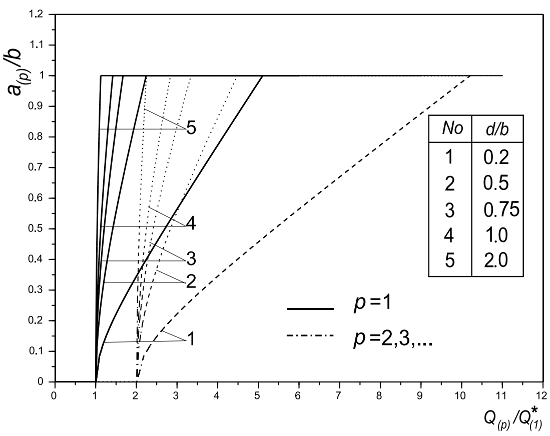

Fig. 2. Dependence of the size of a slippage zone on the loading parameters at the $\boldsymbol{p}$-th step

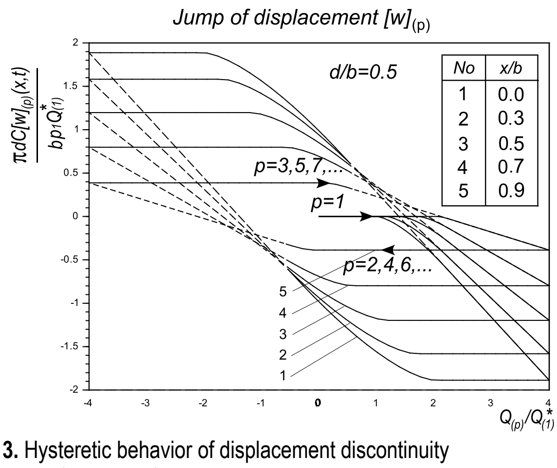

Fig. 3. Hysteretic behavior of displacement discontinuity in a full cycle of loading

Fig. 2 plots the dependence of the size $a_{(p)}/b$ of a slippage zone at the p-th step on the dimensionless magnitude $\beta_{(p)} = Q_{(p)}(t)/Q^*_{(1)}$ of the applied force.

It should be notices that for $\beta_{(p)} \leq 1$ the slippage is always absent, and for $1 \leq \beta_{(p)} \leq 2$ it occurs only at the first (initial) step of a cycle. The slippage zone at a step grows monotonously with increase in the magnitude of loading, not exceeding the size of a crack.

Fig. 3 illustrates the hysteretic behaviour of the total displacement discontinuity $\pi dC[w]_{(p)}(x,t)/bp_1Q^*_{(1)}$ at various points x/b of the slippage zone depending on the magnitude of loading in the alternating cycle $4Q^*_{(1)} \to -4Q^*_{(1)} \to 4Q^*_{(1)} \to -4Q^*_{(1)} \to$ **. ...** Here it is well observed that such character of change is inherent to displacement discontinuities not only at the centre of slippage zone, but also to all of its points. Continuous lines correspond to a range of change of loading between the first and the second critical values, when the slippage zone has not reached crack tips yet. The dot line denote displacement discontinuities for loading exceeding the second critical force, when at the vicinity of crack tips stress singularity arise.

Change in the form of both local and total displacement discontinuities $\pi dC[w]_{(p)}(x,t)/bp_1Q^*_{(1)}$ in the zero-base cycle $10Q^*_{(1)} \to 0 \to 10Q^*_{(1)} \to 0 \to$.. in its dependence on x/b is plotted in Fig. 4. It is well-noted that after a full cycle of change of loading crack edges do not come back into their initial positions, thus keeping some residual displacement discontinuity, which increase together with a friction coefficient. For $Q_{(p)}(t)/Q^*_{(1)} \leq 2$ the slippage at the second and subsequent steps does not occur.

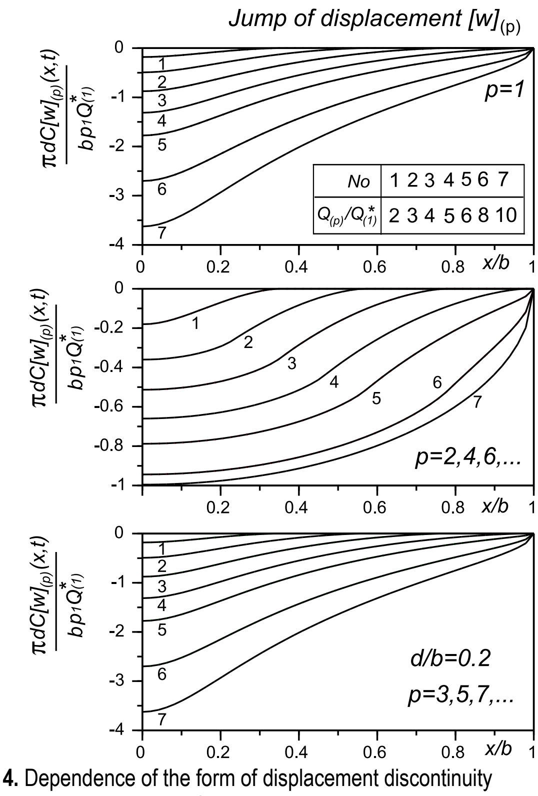

Fig. 4. Dependence of the form of displacement discontinuity on the magnitude of loading at the p-th step

Fig. 5 illustrates the dependence of the form of displacement discontinuity on the relative remoteness d/b of the force application point in the zero-base cycle $4Q^*_{(1)} \to 0 \to 4Q^*_{(1)} \to 0 \to$**. ...** The increase in d/b decreases the range of $Q^{**}_{(p)}/Q^*_{(p)} = \sqrt{d^2+b^2}/d$ and accordingly, the sensitivity of $[w]_{(p)}(x,t)$ to this parameter.

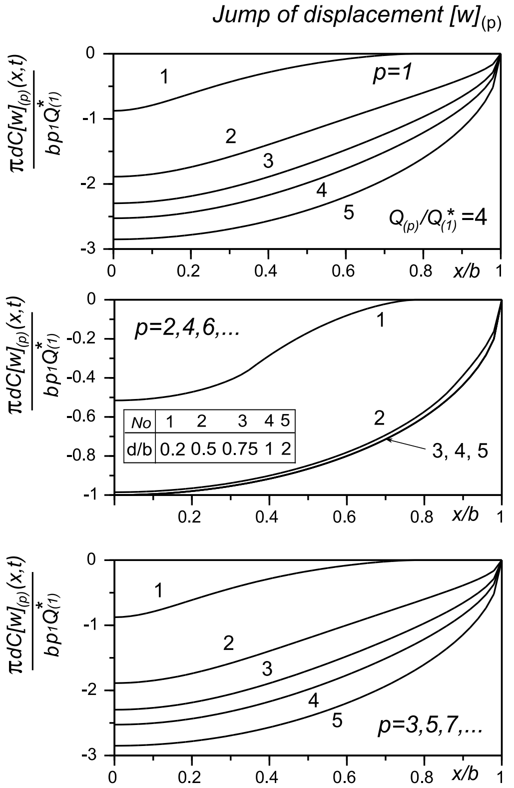

Fig. 5. Dependence of the form of displacement discontinuity on the relative remoteness of the force application point at the p-th step

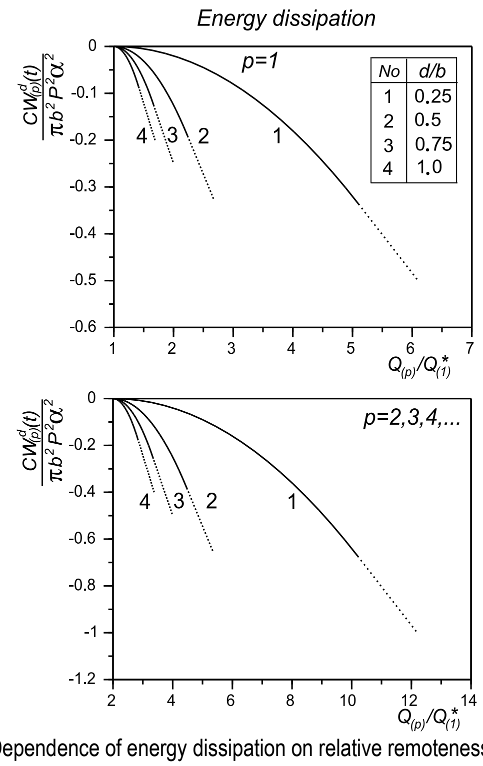

Fig. 6. Dependence of energy dissipation on relative remoteness of the force application point at the p-th step

Energy dispersion $CW_{(p)}^{d}(t)/\pi\alpha^2 b^2 P^2$ at the p-th step of the alternating cycle $4Q_{(1)}^{*} \to -4Q_{(1)}^{*} \to 4Q_{(1)}^{*} \to -4Q_{(1)}^{*} \to \ldots$ depending on the relative remoteness d/b of the force application point is plotted in Fig. 6. Continuous lines correspond to a range of change of loading between the first and the second critical values, when the slippage zone has not reached crack tips yet. The dot line denote displacement discontinuities for loading exceeding the second critical force, when at the vicinity of crack tips stress singularity arise. Total energy dissipation at time t can be obtained using Eq (22).

Thus one can conclude that the parameter G_1/G_2 characterizing the difference of mechanical properties of half-spaces' materials is negligible in the resulted calculations due to a choice of dimensionless quantities of the problem.

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