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ON THE CONVERGENCE OF DOMAIN DECOMPOSITION ALGORITHM FOR THE BODY WITH THIN INCLUSION

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Abstract: We consider a coupled 3D model that involves computation of the stress-strain state for the body with thin inclusion. For the description of the stress-strain state of the main part, the linear elasticity theory is used. The inclusion is modelled using Timoshenko theory for shells. Therefore, the dimension of the problem inside the inclusion is decreased by one. For the numerical solution of this problem we propose an iterative domain decomposition algorithm (Dirichlet-Neumann scheme). This approach allows us to decouple problems in both parts and preserve the structure of the corresponding matrices. We investigate the convergence of the aforementioned algorithm and prove that the problem is well-posed.

Key words: Elasticity Theory, Timoshenko Shell Theory, Steklov-Poincare Operator, Domain Decomposition

1. INTRODUCTION

A lot of structures, that occur in engineering, are inhomogeneous and contain thin parts and massive parts. Therefore, it is important to develop both analytical methods and numerical algorithms for the analysis of the stress-strain state of such structures.

Different aspects of such problems were discussed in Dyyak et al. (2012); Niemi et al. (2010); Savula et al. (2000); Vynnytska and Savula (2008); Nazarov (2005) (in Vynnytska and Savula (2008) the case of the bodies with thin inclusions is considered; in Dyyak et al. (2012) the bodies with thin covers are considered; in Nazarov (2005) asymptotic methods are used for the analysis of the elastic bodies with thin rods). Papers Niemi et al. (2010) and Savula et al. (2000) are devoted to the numerical solution of the Girkmann problem. The discussion on the problems of thermoelasticity the reader may find in Sulym (2007).

In this article, we consider a model for the description of the stress-strain state for the 3D body with thin inclusion. The main part of the body is modelled using the linear elasticity theory. The thin part is modelled using the Timoshenko theory for shells. In order to numerically solve this problem, we propose an iterative domain decomposition algorithm which connects solutions in both parts using coupling conditions. We prove the convergence of the proposed algorithm and the existence and uniqueness of the solution of the corresponding Steklov-Poincare interface equation.

The application of domain decomposition method allows us to decouple problems in both parts and solve the problems independently in each part. As a result, it is possible to compute the stress-strain state accurately even for small shell thicknesses without having problems with stability issues of the coupled problem.

2. PROBLEM STATEMENT

Let us consider a problem of a stress-strain state of an elastic body Ω_1 with the inclusion in Ω_2 (Fig. 1).

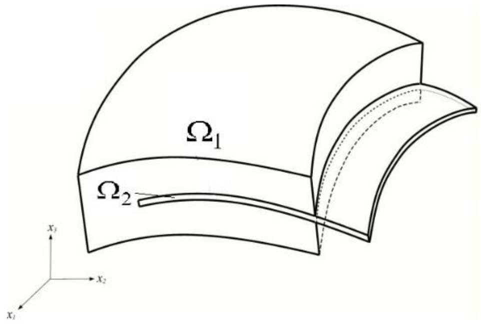

Fig. 1. Body with inclusion

Let us describe the stress-strain state of the body in Ω_1 in rectangular coordinate system x_1, x_2, x_3 using the theory of linear elasticity. Let us denote by $\Sigma = \left(\sigma_{ij}\right)_{i,j=1}^{3}$ the Cauchy stress tensor. The components of Σ are found from the relationships

$$\sigma_{ij} = \frac{1}{2} E_1 \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), i, j = 1,2,3,$$

where: E_1 is the Young's modulus of the body in Ω_1; $\mathrm{u}(\mathrm{x}) = \left(u_1(x), u_2(x), u_3(x)\right)$ is the displacement vector with u_i being the displacements along the directions of x_i, $i = 1,2,3$.

Equilibrium equations for the body in Ω_1 have the form:

$$\begin{aligned} &\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = f_1, \\ &\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} = f_2, \\ &\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} = f_3, \end{aligned} \tag{1}$$

where: $\mathrm{x} \in \Omega_1$, $\mathrm{f} = (f_1, f_2, f_3)$ is the vector of volume forces.

In the following we assume that no volume forces act on the body in Ω_1.

Let us denote by $\mathrm{n} = (n^1,\ n^2,\ n^3)$ outer normal vector and by $\tau_1 = (\tau_1^1,\ \tau_1^2,\ \tau_1^3)$, $\tau_2 = (\tau_2^1,\ \tau_2^2,\ \tau_2^3)$ corresponding tangent vectors.

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Equations (1) are supplemented by the boundary conditions of one of the following types.

Kinematic (Dirichlet) boundary conditions are of the form:

$$u_n = u_n^0, u_{\tau_1} = u_{\tau_1}^0, u_{\tau_2} = u_{\tau_2}^0, \mathrm{x} \in \Gamma_1, \quad (2)$$

where: Γ_1 is the outer boundary of Ω_1; u_n, u_{τ_1} and u_{τ_2} are the components of the displacement vector in the coordinate system $\mathrm{n}, \tau_1, \tau_2$; u_n^0, $u_{\tau_1}^0$ and $u_{\tau_2}^0$ are the prescribed displacements on Γ_1.

Static (Neumann) boundary conditions have the form:

$$\sigma_{nn} = \sigma_{nn}^0, \sigma_{n\tau_1} = \sigma_{n\tau_1}^0, \sigma_{n\tau_2} = \sigma_{n\tau_2}^0, \mathrm{x} \in \Gamma_1, \quad (3)$$

where: σ_{nn}, $\sigma_{n\tau_1}$ and $\sigma_{n\tau_2}$ are the components of the stress tensor in the coordinate system; $\mathrm{n}, \tau_1, \tau_2$; σ_{nn}^0, $\sigma_{n\tau_1}^0$ and $\sigma_{n\tau_2}^0$ are the prescribed stresses on Γ_1.

It is possible to consider other types of boundary conditions, for example mixed boundary conditions, that combine boundary conditions (2) and (3).

For the description of the stress-strain state of the inclusion Ω_2 we use the equations of Timoshenko shell theory in the curvilinear coordinate system (ξ_1, ξ_2, ξ_3) that hold on the median surface Ω_2^*, where: $\Omega_2 = \left\{(\xi_1, \xi_2, \xi_3): \xi_1^b \le \xi_1 \le \xi_1^e, \xi_2^b \le \xi_2 \le \xi_2^e, -\frac{h}{2} \le \xi_3 \le \frac{h}{2}\right\}$, h is the thickness of the inclusion in Ω_2.

By Ω_2^* we denote the median surface of Ω_2 (the projection of Ω_2 on the surface for which $\xi_3 = 0$).

The equations of Timoshenko shell theory are of the form (Pelekh, 1978):

$$\frac{1}{A_1 A_2}\frac{\partial(A_2 T_{11})}{\partial \xi_1} - \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \xi_1} T_{22} + \frac{1}{A_1^2 A_2}\frac{\partial(A_1^2 T_{12})}{\partial \xi_2} +$$
$$+k_1 T_{13} + \frac{1}{A_1 A_2}\frac{\partial\left(\frac{A_1}{r_1} M_{11}\right)}{\partial \xi_2} + \frac{k_2}{A_1 A_2}\frac{\partial A_1}{\partial \xi_2} M_{12} =$$
$$= -(p_1^+ + p_1^-),$$
$$-\frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \xi_2} T_{11} + \frac{1}{A_1 A_2}\frac{\partial(A_1 T_{22})}{\partial \xi_2} + \frac{1}{A_1 A_2^2}\frac{\partial(A_2^2 T_{12})}{\partial \xi_1} +$$
$$+k_2 T_{23} + \frac{1}{A_1 A_2}\frac{\partial\left(\frac{A_2}{r_2} M_{22}\right)}{\partial \xi_2} + \frac{k_1}{A_1 A_2}\frac{\partial A_2}{\partial \xi_1} M_{12} =$$
$$= -(p_2^+ + p_2^-),$$
$$-k_1 T_{11} - k_2 T_{22} + \frac{1}{A_1 A_2}\frac{\partial(A_2 T_{13})}{\partial \xi_1} + \frac{1}{A_1 A_2}\frac{\partial(A_1 T_{23})}{\partial \xi_2} =$$
$$= -(p_3^+ - p_3^-),$$
$$-T_{13} + \frac{1}{A_1 A_2}\frac{\partial(A_2 M_{11})}{\partial \xi_1} - \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \xi_1} M_{22} +$$
$$+\frac{1}{A_1^2 A_2}\frac{\partial(A_1^2 M_{12})}{\partial \xi_2} = -\frac{h}{2}(p_1^+ - p_1^-),$$
$$-T_{23} - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \xi_2} M_{11} + \frac{1}{A_1 A_2}\frac{\partial(A_1 M_{22})}{\partial \xi_2} +$$
$$+\frac{1}{A_1 A_2^2}\frac{\partial(A_2^2 M_{12})}{\partial \xi_1} = -\frac{h}{2}(p_2^+ - p_2^-),$$

where: $T_{11}, T_{12}, T_{22}, T_{13}, T_{23}, M_{11}, M_{12}, M_{22}$ are the forces and momenta in the shell; $A_1 = A_1(\xi_1, \xi_2)$, $A_2 = A_2(\xi_1, \xi_2)$, $k_1 = k_1(\xi_1, \xi_2)$, $k_2 = k_2(\xi_1, \xi_2)$ correspond to Lame parameters and median surface curvature parameters; $r_1 = \frac{1}{k_1}$, $r_2 = \frac{1}{k_2}$; $(\xi_1, \xi_2) \in \Omega_2^*$; p_1^+, p_1^-, p_2^+, p_2^-, p_3^+, p_3^- are given functions; it holds:

$$T_{\alpha\alpha} = \frac{E_2 h}{(1-\vartheta_2^2)}\left(\varepsilon_{\alpha\alpha} + \vartheta_2 \varepsilon_{\beta\beta}\right); \; T_{12} = \frac{E_2 h}{2(1+\vartheta_2)}\varepsilon_{12}$$
$$T_{\alpha 3} = k' G' h \varepsilon_{\alpha 3};$$
$$M_{\alpha\alpha} = \frac{E_2 h^3}{12(1-\vartheta_2^2)}\left(\chi_{\alpha\alpha} + \vartheta_2 \chi_{\beta\beta}\right); M_{12} = \frac{E_2 h^3}{12(1+\vartheta_2)}\chi_{12}$$

where: $\alpha, \beta = 1,2$, $\alpha \ne \beta$; k', G' are constants that characterize transversely isotropic material; E_2 is the Young's modulus of the shell, ϑ_2 is the Poisson's ratio.

The strains ε_{11}, ε_{22}, ε_{12}, ε_{13}, ε_{23}, χ_{11}, χ_{22}, χ_{12} are obtained from the relationships:

$$\varepsilon_{\alpha\alpha} = \frac{1}{A_\alpha}\frac{\partial \mathrm{v}_\alpha}{\partial \xi_\alpha} + \frac{1}{A_\alpha A_\beta}\mathrm{v}_\beta \frac{\partial \mathrm{A}_\alpha}{\partial \xi_\beta} + k_\alpha w,$$

$$2\varepsilon_{\alpha\beta} = \frac{A_\alpha}{A_\beta}\frac{\partial \frac{\mathrm{v}_\alpha}{A_\alpha}}{\partial \xi_\beta} + \frac{A_\beta}{A_\alpha}\frac{\partial \frac{\mathrm{v}_\beta}{A_\beta}}{\partial \xi_\alpha}; \; \varepsilon_{\alpha 3} = -k_\alpha \mathrm{v}_\alpha + \frac{1}{A_\alpha}\frac{\partial w}{\partial \xi_\alpha} + \gamma_\alpha$$

$$\chi_{\alpha\alpha} = \frac{1}{A_\alpha}\frac{\partial \gamma_\alpha}{\partial \xi_\alpha} + \frac{1}{A_\alpha A_\beta}\gamma_\beta \frac{\partial \mathrm{A}_\alpha}{\partial \xi_\beta}$$

$$2\chi_{\alpha\beta} = \frac{k_\alpha}{A_\beta}\frac{\partial \mathrm{v}_\alpha}{\partial \xi_\beta} - \frac{k_\beta}{A_\alpha A_\beta}\mathrm{v}_\alpha \frac{\partial A_\alpha}{\partial \xi_\beta} + \frac{k_\beta}{A_\alpha}\frac{\partial \mathrm{v}_\beta}{\partial \xi_\alpha} -$$

$$-\frac{k_\alpha}{A_\alpha A_\beta}\mathrm{v}_\beta \frac{\partial A_\beta}{\partial \xi_\alpha} + \frac{A_\alpha}{A_\beta}\frac{\partial \frac{\gamma_\alpha}{A_\alpha}}{\partial \xi_\beta} + \frac{A_\beta}{A_\alpha}\frac{\partial \frac{\gamma_\beta}{A_\beta}}{\partial \xi_\alpha}$$

where: $\alpha, \beta = 1,2$, $\alpha \ne \beta$; $\mathrm{v}_1 = \mathrm{v}_1(\xi_1, \xi_2)$, $\mathrm{v}_2 = \mathrm{v}_2(\xi_1, \xi_2)$, $w = w(\xi_1, \xi_2)$, $\gamma_1 = \gamma_1(\xi_1, \xi_2)$, $\gamma_2 = \gamma_2(\xi_1, \xi_2)$ are the displacements and angles of revolution in the shell;

$$p_i^+ = \left(1 + k_1 \frac{h}{2}\right)\left(1 + k_2 \frac{h}{2}\right)\sigma_{i3}^+,$$
$$p_i^- = \left(1 - k_1 \frac{h}{2}\right)\left(1 - k_2 \frac{h}{2}\right)\sigma_{i3}^-, \; i = 1,2,3.$$

Here σ_{i3}^+, σ_{i3}^-, $i = 1,2,3$ are the components of the stress tensor on the top ($\xi_3 = \frac{h}{2}$) and bottom ($\xi_3 = -\frac{h}{2}$) surfaces of the shell. It is known that in the case of isotropic bodies we have $k' = \frac{5}{6}$, $G' = \frac{E_2}{2(1+\vartheta_2)}$.

On the outer edge of the thin part we impose boundary conditions either on the displacements v_1, v_2, w and angles γ_1, γ_2 or on the forces $T_{11}, T_{22}, T_{13}, T_{23}$ and momenta M_{11}, M_{22} in the shell (depending if the corresponding parts of the boundary are subjected to load or free). At the outer surface of the shell we prescribe to σ_{i3}^+ and σ_{i3}^-, $i = 1,2,3$ some given stresses.

The operator form of the equations of Timoshenko shell theory is:

$$\mathrm{Ly} = \mathrm{g}, \quad (4)$$

with:

$$\mathrm{g} = A_1 A_2 (g_1, g_2, g_3, g_4, g_5)^T$$
$$g_1 = p_1^+ + p_1^-; \; g_2 = p_2^+ + p_2^-; \; g_3 = p_3^+ - p_3^-;$$
$$g_4 = \frac{h}{2}(p_1^+ - p_1^-); \; g_5 = \frac{h}{2}(p_2^+ - p_2^-)$$
$$\mathrm{y} = (\mathrm{v}_1, \mathrm{v}_2, w, \gamma_1, \gamma_2)^T, \; \mathrm{Ly} = (l_1, l_2, l_3, l_4, l_5)^T$$
$$l_1 = -\frac{\partial(A_2 T_{11})}{\partial \xi_1} + \frac{\partial A_2}{\partial \xi_1} T_{22} - \frac{1}{A_1}\frac{\partial(A_1^2 T_{12})}{\partial \xi_2} -$$
$$-A_1 A_2 k_1 T_{13} - \frac{\partial\left(\frac{A_1}{r_1} M_{11}\right)}{\partial \xi_2} - k_2 \frac{\partial A_1}{\partial \xi_2} M_{12}$$

$$l_2 = \frac{\partial A_1}{\partial \xi_2} T_{11} - \frac{\partial (A_1 T_{22})}{\partial \xi_2} - \frac{1}{A_2} \frac{\partial (A_2^2 T_{12})}{\partial \xi_1} -$$

$$-A_1 A_2 k_2 T_{23} - \frac{\partial \left(\frac{A_2}{r_2} M_{22} \right)}{\partial \xi_2} - k_1 \frac{\partial A_2}{\partial \xi_1} M_{12}$$

$$l_3 = A_1 A_2 k_1 T_{11} + A_1 A_2 k_2 T_{22} - \frac{\partial (A_2 T_{13})}{\partial \xi_1} - \frac{\partial (A_1 T_{23})}{\partial \xi_2}$$

$$l_4 = A_1 A_2 T_{13} - \frac{\partial (A_2 M_{11})}{\partial \xi_1} + \frac{\partial A_2}{\partial \xi_1} M_{22} - \frac{1}{A_1} \frac{\partial (A_1^2 M_{12})}{\partial \xi_2}$$

$$l_5 = A_1 A_2 T_{23} + \frac{\partial A_1}{\partial \xi_2} M_{11} - \frac{\partial (A_1 M_{22})}{\partial \xi_2} - \frac{1}{A_2} \frac{\partial (A_2^2 M_{12})}{\partial \xi_1}$$

Let us write down the weak formulation of the Timoshenko shell theory problem. Without loss of generality we assume homogeneous boundary conditions. Let us define the following function spaces: $V = \{v \in H^1(\Omega_2^*) \colon v = 0,\ \xi = (\xi_1, \xi_2, \xi_3) \in \Gamma_{2D}\}$, $V_1 = V^5$, where Γ_{2D} is the part of the outer boundary of Ω_2 on which the kinematic boundary condition is prescribed.

The weak formulation of the problem (4) has the form: find $\mathrm{y} \in V_1$, such that:

$$a(\mathrm{y}, \mathrm{v}) = g(\mathrm{v}),\ \forall \mathrm{v} \in V_1, \tag{5}$$

where: $a(\mathrm{y}, \mathrm{v}) = (\mathrm{Ly}, \mathrm{v})$, $g(\mathrm{v}) = (\mathrm{g}, \mathrm{v})$.

Lemma. Assume that for the problem (4) there exist positive constants $r_{10}, r_{20}, A_{10}, A_{20}$ such that:

1. $|r_1| \geq r_{10} > 0$; $|r_2| \geq r_{20} > 0$;
2. $|A_1| \geq A_{10} > 0$; $|A_2| \geq A_{20} > 0$

almost everywhere in Ω_2^*.

Then the bilinear form $a(\mathrm{y}, \mathrm{v})$ for the problem (5) of Timoshenko shell theory is continuous.

Proof. Firstly, we remark that from the assumption 1) it follows that $|k_1| \leq k_{10} < \infty$, $|k_2| \leq k_{20} < \infty$. Moreover, the assumptions of the lemma do not restrict the class of the problems or the algorithm that can be used for the numerical solution of this problem. Indeed, for the points, for which the assumptions do not hold, the system becomes singular and can no longer be used for the adequate description of the physical process that is being modeled.

The continuity of the bilinear form follows from the fact, that all the coefficients in the system (4) are bounded by modulus almost everywhere, and that the system (4) itself is linear.

The proof also uses the obvious inequality:

$ab \leq \frac{a^2+b^2}{2}$ for $a, b \in R$.

In order to couple the models in both parts, adequate boundary conditions need to be specified. Let us denote by Ω_{2in} part of the inclusion that lies inside the body Ω_1 and by Ω_{2in}^* part of the median surface Ω_2^* which is the projection of Ω_{2in} on the surface $\xi_3 = 0$.

Let Γ_I be a boundary, common to both Ω_1 and Ω_2. Let us divide Γ_I into the following parts:

$$\Gamma_{I_1} = \left\{ \xi = (\xi_1, \xi_2, \xi_3) \colon (\xi_1, \xi_2) \in \Omega_{2in}^*;\ \xi_3 = -\frac{h}{2} \right\}$$

$$\Gamma_{I_2} = \left\{ \begin{array}{c} \xi = (\xi_1, \xi_2, \xi_3) \colon (\xi_1, \xi_2) \in \Omega_{2in}^*; \\ \xi_1 = \xi_1^b;\ -\frac{h}{2} \leq \xi_3 \leq \frac{h}{2} \end{array} \right\}$$

$$\Gamma_{I_3} = \left\{ \begin{array}{c} \xi = (\xi_1, \xi_2, \xi_3) \colon (\xi_1, \xi_2) \in \Omega_{2in}^*; \\ \xi_2 = \xi_2^b;\ -\frac{h}{2} \leq \xi_3 \leq \frac{h}{2} \end{array} \right\}$$

$$\Gamma_{I_4} = \left\{ \xi = (\xi_1, \xi_2, \xi_3) \colon (\xi_1, \xi_2) \in \Omega_{2in}^*;\ \xi_3 = \frac{h}{2} \right\}$$

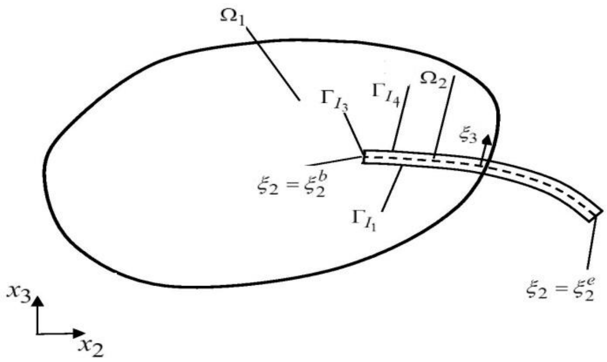

Fig. 2. Cross-section of the body in Ω by the plane $x_1 = \mathrm{const}$.

Fig. 2 shows the cross-section of the body in Ω by the plane $x_1 = \mathrm{const}$ and the connection between rectangular x_1, x_2, x_3 and curvilinear ξ_1, ξ_2, ξ_3 coordinate systems.

On each part of Γ_I the following coupling conditions are prescribed (Pelekh, 1978):

– on Γ_{I_1} (inner part of bottom surface of Ω_2):

$$u_n = w,\ u_{\tau_1} = -\mathrm{v}_1 + \frac{h}{2}\gamma_1,\ u_{\tau_2} = -\mathrm{v}_2 + \frac{h}{2}\gamma_2,$$

$$\sigma_{nn} = -\sigma_{33}^-,\ \sigma_{n\tau_1} = -\sigma_{13}^-,\ \sigma_{n\tau_2} = -\sigma_{23}^-; \tag{6}$$

– on Γ_{I_2} (inner edge of Ω_2):

$$u_{\tau_1} = w,\ u_{\tau_2} = -\mathrm{v}_2 - \xi_3 \gamma_2,\ u_n = \mathrm{v}_1 + \xi_3 \gamma_1,$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{nn} d\xi_3 = T_{11},\quad \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{n\tau_1} d\xi_3 = T_{13},\quad \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{n\tau_2} d\xi_3 = T_{12},$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{nn} \xi_3 d\xi_3 = M_{11},\ \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{n\tau_2} \xi_3 d\xi_3 = M_{12}; \tag{7}$$

– on Γ_{I_3} (inner edge of Ω_2):

$$u_{\tau_2} = w,\ u_{\tau_1} = -\mathrm{v}_1 - \xi_3 \gamma_1,\ u_n = \mathrm{v}_2 + \xi_3 \gamma_2,$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{nn} d\xi_3 = T_{22},\quad \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{n\tau_2} d\xi_3 = T_{23},\quad \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{n\tau_1} d\xi_3 = T_{12},$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{nn} \xi_3 d\xi_3 = M_{22},\ \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{n\tau_1} \xi_3 d\xi_3 = M_{12}; \tag{8}$$

– on Γ_{I_4} (inner part of top surface surface of Ω_2):

$$u_n = -w,\ u_{\tau_1} = \mathrm{v}_1 + \frac{h}{2}\gamma_1,\ u_{\tau_2} = \mathrm{v}_2 + \frac{h}{2}\gamma_2,$$

$$\sigma_{nn} = -\sigma_{33}^+,\ \sigma_{n\tau_1} = \sigma_{13}^+,\ \sigma_{n\tau_2} = \sigma_{23}^+. \tag{9}$$

3. DOMAIN DECOMPOSITION ALGORITHM

For the numerical solution of the model domain decomposition algorithm can be used.

The approximate solutions in both domains are connected using Dirichlet-Neumann scheme (Quarteroni and Valli, 1999). Do-

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main decomposition algorithm has the following form:

1. set an initial guess λ^0 for the unknown displacements on the interface Γ_I, set $\varepsilon > 0$;
2. for $k = 0,1, \ldots$ solve the boundary value problem in Ω_1 with the displacements equal to λ^k to obtain the approximation for the forces and momenta in Ω_2 using (6)-(9);
3. solve the corresponding problem in Ω_2 to find the displacements $u_n^k, u_{\tau_1}^k, u_{\tau_2}^k$ on Γ_I;
4. update the displacements λ^k on Γ_I:
   – on Γ_{I_1}:

$$\lambda_{11}^{k+1} = (1-\theta)\lambda_{11}^k + \theta u_n^k, \qquad \lambda_{12}^{k+1} = (1-\theta)\lambda_{12}^k + \theta u_{\tau_1}^k,$$
$$\lambda_{13}^{k+1} = (1-\theta)\lambda_{13}^k + \theta u_{\tau_2}^k;$$

   – on Γ_{I_2}:

$$\lambda_{21}^{k+1} = (1-\theta)\lambda_{21}^k + \theta \mathrm{v}_1^k, \qquad \lambda_{22}^{k+1} = (1-\theta)\lambda_{22}^k + \theta \mathrm{v}_2^k,$$
$$\lambda_{23}^{k+1} = (1-\theta)\lambda_{23}^k + \theta w^k, \qquad \lambda_{24}^{k+1} = (1-\theta)\lambda_{24}^k + \theta \gamma_1^k,$$
$$\lambda_{25}^{k+1} = (1-\theta)\lambda_{25}^k + \theta \gamma_2^k;$$

   – on Γ_{I_3}:

$$\lambda_{31}^{k+1} = (1-\theta)\lambda_{31}^k + \theta \mathrm{v}_2^k, \qquad \lambda_{32}^{k+1} = (1-\theta)\lambda_{32}^k + \theta \mathrm{v}_1^k,$$
$$\lambda_{33}^{k+1} = (1-\theta)\lambda_{33}^k + \theta w^k, \qquad \lambda_{34}^{k+1} = (1-\theta)\lambda_{34}^k + \theta \gamma_2^k,$$
$$\lambda_{35}^{k+1} = (1-\theta)\lambda_{35}^k + \theta \gamma_1^k;$$

   – on Γ_{I_4}:

$$\lambda_{41}^{k+1} = (1-\theta)\lambda_{41}^k + \theta u_n^k,$$
$$\lambda_{42}^{k+1} = (1-\theta)\lambda_{42}^k + \theta u_{\tau_1}^k,$$
$$\lambda_{43}^{k+1} = (1-\theta)\lambda_{43}^k + \theta u_{\tau_2}^k,$$

   where $\theta > 0$ is a relaxation parameter;
5. if $\|\lambda^{k+1} - \lambda^k\| \geq \varepsilon$, then go to step 2, otherwise the algorithm ends.

In the following we assume that the variational problems corresponding to the domains Ω_1, Ω_2 and Ω have unique solutions (Hsiao and Wendland, 2008; Vynnytska and Savula, 2012; Dyyak and Savula, 1997).

Let us prove the convergence of the domain decomposition algorithm and the existence and uniqueness of the solution of the corresponding Steklov-Poincare equation.

For this purpose let us introduce on the common boundary of both domains the function $\varphi \in \Lambda$, with:

$$\Lambda = \{\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)\},$$

$$\varphi_1 = (\varphi_{11}, \varphi_{12}, \varphi_{13}), \varphi_{1i} \in H^1(\Gamma_{I_1}), i = 1,2,3;$$

$$\varphi_2 = (\varphi_{21}, \varphi_{22}, \varphi_{23}, \varphi_{24}, \varphi_{25}),$$

$$\varphi_{2i} = \varphi_{2i}(\xi_2) \in H^{\frac{1}{2}}(\Gamma_{I_2}), \qquad i = 1,2,3,4,5;$$

$$\varphi_3 = (\varphi_{31}, \varphi_{32}, \varphi_{33}, \varphi_{34}, \varphi_{35}),$$

$$\varphi_{3i} = \varphi_{3i}(\xi_1) \in H^{\frac{1}{2}}(\Gamma_{I_3}), \qquad i = 1,2,3,4,5;$$

$$\varphi_4 = (\varphi_{41}, \varphi_{42}, \varphi_{43}), \varphi_{4i} \in H^1(\Gamma_{I_4}), i = 1,2,3.$$

We remark, that the choice of the space Λ is motivated by the specifics of the model and is based on the regularity of the corresponding functions on each part of the interface Γ_I.

The connection between the functions φ_{ij} and the displacements on the interface is the following:

– on Γ_{I_1}:

$$\varphi_{11} = u_n, \varphi_{12} = u_{\tau_1}, \varphi_{13} = u_{\tau_2};$$

– on Γ_{I_2}:

$$\varphi_{21} = \mathrm{v}_1, \varphi_{22} = \mathrm{v}_2, \varphi_{23} = w, \varphi_{24} = \gamma_1, \varphi_{25} = \gamma_2;$$

– on Γ_{I_3}:

$$\varphi_{31} = \mathrm{v}_2, \varphi_{32} = \mathrm{v}_1, \varphi_{33} = w, \varphi_{34} = \gamma_2, \varphi_{35} = \gamma_1;$$

– on Γ_{I_4}:

$$\varphi_{41} = u_n, \varphi_{42} = u_{\tau_1}, \varphi_{43} = u_{\tau_2}.$$

Let S be a Steklov-Poincare operator for our problem and $S_i, i = 1,2$ be local Steklov-Poincare operators corresponding to the domains Ω_i. Steklov-Poincare operator for the boundary-value problem is an operator that transforms boundary conditions of one type into boundary conditions of another type. In our case, Steklov-Poincare operator transforms the displacements on the boundary into loads on the boundary.

Let us multiply interface conditions (6) by $A_1 A_2 \left(1 - k_1 \frac{h}{2}\right)\left(1 - k_2 \frac{h}{2}\right)$; (7) and (8) – by $\frac{1}{h}$; (9) – by $A_1 A_2 \left(1 + k_1 \frac{h}{2}\right)\left(1 + k_2 \frac{h}{2}\right)$.

The Steklov-Poincare operator can be written in the form:

$\{S\varphi, \psi\}_{\Gamma_I} = \{S_1\varphi, \psi\}_{\Gamma_I} + \{S_2\varphi, \psi\}_{\Gamma_I}$, where:

$$\begin{aligned}
\{S_1\varphi, \psi\}_{\Gamma_I} &= \\
&= \langle -A_1A_2\left(1 - k_1\tfrac{h}{2}\right)\left(1 - k_2\tfrac{h}{2}\right)\sigma_{nn}(\varphi), \psi_{11}\rangle_{\Gamma_{I_1}} + \\
&+ \langle -A_1A_2\left(1 - k_1\tfrac{h}{2}\right)\left(1 - k_2\tfrac{h}{2}\right)\sigma_{n\tau_1}(\varphi), \psi_{12}\rangle_{\Gamma_{I_1}} + \\
&+ \langle -A_1A_2\left(1 - k_1\tfrac{h}{2}\right)\left(1 - k_2\tfrac{h}{2}\right)\sigma_{n\tau_2}(\varphi), \psi_{13}\rangle_{\Gamma_{I_1}} + \\
&+ \langle -A_1A_2\left(1 + k_1\tfrac{h}{2}\right)\left(1 + k_2\tfrac{h}{2}\right)\sigma_{nn}(\varphi), \psi_{41}\rangle_{\Gamma_{I_4}} + \\
&+ \langle -A_1A_2\left(1 + k_1\tfrac{h}{2}\right)\left(1 + k_2\tfrac{h}{2}\right)\sigma_{n\tau_1}(\varphi), \psi_{42}\rangle_{\Gamma_{I_4}} + \\
&+ \langle -A_1A_2\left(1 + k_1\tfrac{h}{2}\right)\left(1 + k_2\tfrac{h}{2}\right)\sigma_{n\tau_2}(\varphi), \psi_{43}\rangle_{\Gamma_{I_4}} + \\
&+ \langle -\tfrac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{nn}(\varphi)d\xi_3, \psi_{21}\rangle_{\Gamma_{I_2}} + \\
&+ \langle -\tfrac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{n\tau_2}(\varphi)d\xi_3, \psi_{22}\rangle_{\Gamma_{I_2}} + \\
&+ \langle -\tfrac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{n\tau_1}(\varphi)d\xi_3, \psi_{23}\rangle_{\Gamma_{I_2}} + \\
&+ \langle -\tfrac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{nn}(\varphi)\xi_3 d\xi_3, \psi_{24}\rangle_{\Gamma_{I_2}} + \\
&+ \langle -\tfrac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{n\tau_2}(\varphi)\xi_3 d\xi_3, \psi_{25}\rangle_{\Gamma_{I_2}} + \\
&+ \langle -\tfrac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{nn}(\varphi)d\xi_3, \psi_{31}\rangle_{\Gamma_{I_3}} + \\
&+ \langle -\tfrac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{n\tau_1}(\varphi)d\xi_3, \psi_{32}\rangle_{\Gamma_{I_3}} + \\
&+ \langle -\tfrac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{n\tau_2}(\varphi)d\xi_3, \psi_{33}\rangle_{\Gamma_{I_3}} + \\
&+ \langle -\tfrac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{nn}(\varphi)\xi_3 d\xi_3, \psi_{34}\rangle_{\Gamma_{I_3}} + \\
&+ \langle -\tfrac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{n\tau_1}(\varphi)\xi_3 d\xi_3, \psi_{35}\rangle_{\Gamma_{I_3}},
\end{aligned}$$

$$
\begin{aligned}
&\{S_2\varphi,\psi\}_{\Gamma_I} = \\
&= \langle -A_1A_2\left(1-k_1\tfrac{h}{2}\right)\left(1-k_2\tfrac{h}{2}\right)\sigma_{33}^{-},\psi_{11}\rangle_{\Gamma_{I_1}} + \\
&+ \langle -A_1A_2\left(1-k_1\tfrac{h}{2}\right)\left(1-k_2\tfrac{h}{2}\right)\sigma_{13}^{-},\psi_{12}\rangle_{\Gamma_{I_1}} + \\
&+ \langle -A_1A_2\left(1-k_1\tfrac{h}{2}\right)\left(1-k_2\tfrac{h}{2}\right)\sigma_{23}^{-},\psi_{13}\rangle_{\Gamma_{I_1}} + \\
&+ \langle -A_1A_2\left(1+k_1\tfrac{h}{2}\right)\left(1+k_2\tfrac{h}{2}\right)\sigma_{33}^{+},\psi_{41}\rangle_{\Gamma_{I_4}} + \\
&+ \langle A_1A_2\left(1+k_1\tfrac{h}{2}\right)\left(1+k_2\tfrac{h}{2}\right)\sigma_{13}^{+},\psi_{42}\rangle_{\Gamma_{I_4}} + \langle A_1A_2\Big(1+ \\
&+k_1\tfrac{h}{2}\Big)\left(1+k_2\tfrac{h}{2}\right)\sigma_{23}^{+},\psi_{43}\rangle_{\Gamma_{I_4}} + \langle \tfrac{1}{h}T_{11},\psi_{21}\rangle_{\Gamma_{I_2}} + \\
&+ \langle \tfrac{1}{h}T_{12},\psi_{22}\rangle_{\Gamma_{I_2}} + \langle \tfrac{1}{h}T_{13},\psi_{23}\rangle_{\Gamma_{I_2}} + \langle \tfrac{1}{h}M_{11},\psi_{24}\rangle_{\Gamma_{I_2}} + \\
&+ \langle \tfrac{1}{h}M_{12},\psi_{25}\rangle_{\Gamma_{I_2}} + \langle \tfrac{1}{h}T_{22},\psi_{31}\rangle_{\Gamma_{I_3}} + \langle \tfrac{1}{h}T_{12},\psi_{32}\rangle_{\Gamma_{I_3}} + \\
&+ \langle \tfrac{1}{h}T_{23},\psi_{33}\rangle_{\Gamma_{I_3}} + \langle \tfrac{1}{h}M_{22},\psi_{34}\rangle_{\Gamma_{I_3}} + \langle \tfrac{1}{h}M_{12},\psi_{35}\rangle_{\Gamma_{I_3}}
\end{aligned}
$$

with $\langle u,v\rangle_{\Gamma_I}$ being a bilinear form:

$$\langle u,v\rangle_{\Gamma_I} = \int_{\Gamma_I} uv d\Gamma_I,\ \forall v \in H^{1/2}(\Gamma_I),\ \forall u \in \left(H^{\frac{1}{2}}(\Gamma_I)\right)^{*}.$$

Let Q, Q_1, Q_2 be preconditioners of the domain decomposition algorithm for the Dirichlet-Neumann scheme (Quarteroni and Valli, 1999), where: $\mathrm{Q} = \mathrm{Q}_1 + \mathrm{Q}_2$, $\{\mathrm{Q}_1\varphi,\psi\}_{\Gamma_I} = \{S_1\varphi,\psi\}_{\Gamma_I}$, $\{\mathrm{Q}_2\varphi,\psi\}_{\Gamma_I} = \{S_2\varphi,\psi\}_{\Gamma_I}$.

In the case of Dirichlet-Neumann scheme the preconditioners Q, Q_1 and Q_2 coincide with Stelkov-Poincare operators S, S_1 and S_2 respectively.

Let us investigate the properties of the Steklov-Poincare operators S, S_1, S_2.

The linearity and symmetry of S_2 follows directly from the linearity of the corresponding operator in Ω_2^*, median surface of Ω_2.

Theorem. Operator S_2 is continuous and positive–definite on Λ.

Proof. Let us rewrite operator S_2 in the form:

$$
\begin{aligned}
&\{S_2\varphi,\psi\}_{\Gamma_I} = \\
&= \int_{\Omega_2^*} A_1A_2\Big(\left(1+k_1\tfrac{h}{2}\right)\left(1+k_2\tfrac{h}{2}\right)\sigma_{33}^{+} - \left(1-k_1\tfrac{h}{2}\right)\Big(1- \\
&-k_2\tfrac{h}{2}\Big)\sigma_{33}^{-}\Big)\tilde{w}\,d\Omega_2^* + \int_{\Omega_2^*} A_1A_2\Big(\left(1+k_1\tfrac{h}{2}\right)\Big(1+ \\
&+k_2\tfrac{h}{2}\Big)\sigma_{13}^{+} + \left(1-k_1\tfrac{h}{2}\right)\left(1-k_2\tfrac{h}{2}\right)\sigma_{13}^{-}\Big)\tilde{v}_1 d\Omega_2^* + \\
&+ \int_{\Omega_2^*} A_1A_2\tfrac{h}{2}\Big(\left(1+k_1\tfrac{h}{2}\right)\left(1+k_2\tfrac{h}{2}\right)\sigma_{13}^{+} - \left(1-k_1\tfrac{h}{2}\right)\Big(1- \\
&-k_2\tfrac{h}{2}\Big)\sigma_{13}^{-}\Big)\tilde{\gamma}_1 d\Omega_2^* + \int_{\Omega_2^*} A_1A_2\Big(\left(1+k_1\tfrac{h}{2}\right)\Big(1+ \\
&+k_2\tfrac{h}{2}\Big)\sigma_{23}^{+} + \left(1-k_1\tfrac{h}{2}\right)\left(1-k_2\tfrac{h}{2}\right)\sigma_{23}^{-}\Big)\tilde{v}_2 d\Omega_2^* + \\
&+ \int_{\Omega_2^*} A_1A_2\tfrac{h}{2}\Big(\left(1+k_1\tfrac{h}{2}\right)\left(1+k_2\tfrac{h}{2}\right)\sigma_{23}^{+} - \left(1-k_1\tfrac{h}{2}\right)\Big(1- \\
&-k_2\tfrac{h}{2}\Big)\sigma_{23}^{-}\Big)\tilde{\gamma}_2 d\Omega_2^* + \langle \tfrac{1}{h}T_{11},\tilde{v}_1\rangle_{\Gamma_{I_2}} + \langle \tfrac{1}{h}T_{12},\tilde{v}_2\rangle_{\Gamma_{I_2}} + \\
&+ \langle \tfrac{1}{h}T_{13},\tilde{w}\rangle_{\Gamma_{I_2}} + \langle \tfrac{1}{h}M_{11},\tilde{\gamma}_1\rangle_{\Gamma_{I_2}} + \langle \tfrac{1}{h}M_{12},\tilde{\gamma}_2\rangle_{\Gamma_{I_2}} + \\
&+ \langle \tfrac{1}{h}T_{22},\tilde{v}_2\rangle_{\Gamma_{I_3}} + \langle \tfrac{1}{h}T_{12},\tilde{v}_1\rangle_{\Gamma_{I_3}} + \langle \tfrac{1}{h}T_{23},\tilde{w}\rangle_{\Gamma_{I_3}} + \\
&+ \langle \tfrac{1}{h}M_{22},\tilde{\gamma}_2\rangle_{\Gamma_{I_3}} + \langle \tfrac{1}{h}M_{12},\tilde{\gamma}_1\rangle_{\Gamma_{I_3}}.
\end{aligned}
$$

Let us substitute the corresponding left-hand sides from the Timoshenko shell theory model equations (4). As a result, we can prove the continuity and coercitivity of the local Steklov-Poincare operator S_2 taking into account properties of the operator (4). It is known, that the operator (4) is coercive (Vynnytska and Savula, 2008).

Therefore, we obtain:

$$
\begin{aligned}
&\{S_2\varphi,\varphi\}_{\Gamma_I} \geq c^2 \int_{\Omega_2^*}\Big(\left(\tfrac{\partial v_1}{\partial\xi_1}\right)^2 + \left(\tfrac{\partial v_2}{\partial\xi_1}\right)^2 + \left(\tfrac{\partial w}{\partial\xi_1}\right)^2 + \left(\tfrac{\partial\gamma_1}{\partial\xi_1}\right)^2 + \\
&+\left(\tfrac{\partial\gamma_2}{\partial\xi_1}\right)^2 + \left(\tfrac{\partial v_1}{\partial\xi_2}\right)^2 + \left(\tfrac{\partial v_2}{\partial\xi_2}\right)^2 + \left(\tfrac{\partial w}{\partial\xi_2}\right)^2 + \left(\tfrac{\partial\gamma_1}{\partial\xi_2}\right)^2 + \left(\tfrac{\partial\gamma_2}{\partial\xi_2}\right)^2 + \\
&+v_1^2 + v_2^2 + w^2 + \gamma_1^2 + \gamma_2^2\Big)\,d\Omega_2^*,\ c \neq 0. \qquad (10)
\end{aligned}
$$

From (10) it follows that:

$$
\begin{aligned}
&\{S_2\varphi,\varphi\}_{\Gamma_I} \geq c_1^2 \int_{\Omega_2^*}\Big(\left(-\tfrac{\partial v_1}{\partial\xi_1} + \tfrac{h}{2}\tfrac{\partial\gamma_1}{\partial\xi_1}\right)^2 + \Big(-\tfrac{\partial v_2}{\partial\xi_1} + \\
&+\tfrac{h}{2}\tfrac{\partial\gamma_2}{\partial\xi_1}\Big)^2 + \left(-\tfrac{\partial v_1}{\partial\xi_2} + \tfrac{h}{2}\tfrac{\partial\gamma_1}{\partial\xi_2}\right)^2 + \left(-\tfrac{\partial v_2}{\partial\xi_2} + \tfrac{h}{2}\tfrac{\partial\gamma_2}{\partial\xi_2}\right)^2 + \\
&+\left(\tfrac{\partial w}{\partial\xi_1}\right)^2 + \left(\tfrac{\partial w}{\partial\xi_2}\right)^2 + \left(-v_1 + \tfrac{h}{2}\gamma_1\right)^2 + \left(-v_2 + \tfrac{h}{2}\gamma_2\right)^2 + \\
&+w^2\Big)\,d\Omega_2^* + c_2^2\|\varphi_2\|^2_{H^{\frac{1}{2}}(\Gamma_{I_2})} + c_3^2\|\varphi_3\|^2_{H^{\frac{1}{2}}(\Gamma_{I_3})} + \\
&+c_4^2 \int_{\Omega_2^*}\Big(\left(\tfrac{\partial v_1}{\partial\xi_1} + \tfrac{h}{2}\tfrac{\partial\gamma_1}{\partial\xi_1}\right)^2 + \left(\tfrac{\partial v_2}{\partial\xi_1} + \tfrac{h}{2}\tfrac{\partial\gamma_2}{\partial\xi_1}\right)^2 + \Big(\tfrac{\partial v_1}{\partial\xi_2} + \\
&+\tfrac{h}{2}\tfrac{\partial\gamma_1}{\partial\xi_2}\Big)^2 + \left(\tfrac{\partial v_2}{\partial\xi_2} + \tfrac{h}{2}\tfrac{\partial\gamma_2}{\partial\xi_2}\right)^2 + \left(\tfrac{\partial w}{\partial\xi_1}\right)^2 + \left(\tfrac{\partial w}{\partial\xi_2}\right)^2 + \\
&+\left(v_1 + \tfrac{h}{2}\gamma_1\right)^2 + \left(v_2 + \tfrac{h}{2}\gamma_2\right)^2 + w^2\Big)\,d\Omega_2^*,
\end{aligned}
$$

$c_i > 0,\ i = 1, 2, 3, 4.$

Therefore, the operator S_2 is coercive on Λ.

Let us prove the continuity of S_2. The continuity of S_2 follows from the continuity of the operator for the problem (4) in Ω_2^*.

Using the continuity of the operator for the problem (4), we get:

$$
\begin{aligned}
&\{S_2\varphi,\psi\}_{\Gamma_I} \leq C^2\Big(\int_{\Omega_2^*}\Big(\left(\tfrac{\partial v_1}{\partial\xi_1}\right)^2 + \left(\tfrac{\partial v_2}{\partial\xi_1}\right)^2 + \left(\tfrac{\partial w}{\partial\xi_1}\right)^2 + \\
&+\left(\tfrac{\partial\gamma_1}{\partial\xi_1}\right)^2 + \left(\tfrac{\partial\gamma_2}{\partial\xi_1}\right)^2 + \left(\tfrac{\partial v_1}{\partial\xi_2}\right)^2 + \left(\tfrac{\partial v_2}{\partial\xi_2}\right)^2 + \left(\tfrac{\partial w}{\partial\xi_2}\right)^2 + \left(\tfrac{\partial\gamma_1}{\partial\xi_2}\right)^2 + \\
&+\left(\tfrac{\partial\gamma_2}{\partial\xi_2}\right)^2 + v_1^2 + v_2^2 + w^2 + \gamma_1^2 + \gamma_2^2\Big)\,d\Omega_2^*\Big)^{1/2} \times \\
&\times\Big(\int_{\Omega_2^*}\Big(\left(\tfrac{\partial\tilde{v}_1}{\partial\xi_1}\right)^2 + \left(\tfrac{\partial\tilde{v}_2}{\partial\xi_1}\right)^2 + \left(\tfrac{\partial\tilde{w}}{\partial\xi_1}\right)^2 + \left(\tfrac{\partial\tilde{\gamma}_1}{\partial\xi_1}\right)^2 + \left(\tfrac{\partial\tilde{\gamma}_2}{\partial\xi_1}\right)^2 + \\
&+\left(\tfrac{\partial\tilde{v}_1}{\partial\xi_2}\right)^2 + \left(\tfrac{\partial\tilde{v}_2}{\partial\xi_2}\right)^2 + \left(\tfrac{\partial\tilde{w}}{\partial\xi_2}\right)^2 + \left(\tfrac{\partial\tilde{\gamma}_1}{\partial\xi_2}\right)^2 + \left(\tfrac{\partial\tilde{\gamma}_2}{\partial\xi_2}\right)^2 + \tilde{v}_1^2 + \\
&+\tilde{v}_2^2 + \tilde{w}^2 + \tilde{\gamma}_1^2 + \tilde{\gamma}_2^2\Big)\,d\Omega_2^*\Big)^{1/2},\ C > 0.
\end{aligned}
$$

As a result, S_2 is continuous.

Let us consider now the local Steklov-Poincare operator S_1 and rewrite it in the form:

$$
\begin{aligned}
&\{S_1\varphi,\psi\}_{\Gamma_I} = \\
&= \langle -A_1A_2\left(1-k_1\tfrac{h}{2}\right)\left(1-k_2\tfrac{h}{2}\right)\sigma_{nn}(\varphi),u_n\rangle_{\Gamma_{I_1}} + \\
&+ \langle -A_1A_2\left(1-k_1\tfrac{h}{2}\right)\left(1-k_2\tfrac{h}{2}\right)\sigma_{n\tau_1}(\varphi),u_{\tau_1}\rangle_{\Gamma_{I_1}} + \\
&+ \langle -A_1A_2\left(1-k_1\tfrac{h}{2}\right)\left(1-k_2\tfrac{h}{2}\right)\sigma_{n\tau_2}(\varphi),u_{\tau_2}\rangle_{\Gamma_{I_1}} + \\
&+ \langle -A_1A_2\left(1+k_1\tfrac{h}{2}\right)\left(1+k_2\tfrac{h}{2}\right)\sigma_{nn}(\varphi),u_n\rangle_{\Gamma_{I_4}} + \\
&+ \langle -A_1A_2\left(1+k_1\tfrac{h}{2}\right)\left(1+k_2\tfrac{h}{2}\right)\sigma_{n\tau_1}(\varphi),u_{\tau_1}\rangle_{\Gamma_{I_4}} + \\
&+ \langle -A_1A_2\left(1+k_1\tfrac{h}{2}\right)\left(1+k_2\tfrac{h}{2}\right)\sigma_{n\tau_2}(\varphi),u_{\tau_2}\rangle_{\Gamma_{I_4}} + \\
&+ \langle -\tfrac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_{nn}(\varphi)d\xi_3\,,u_n\rangle_{\Gamma_{I_2}} + \\
&+ \langle \tfrac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_{n\tau_2}(\varphi)d\xi_3\,,u_{\tau_2}\rangle_{\Gamma_{I_2}} +
\end{aligned}
$$

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$$+\langle -\frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{n\tau_1}(\varphi)d\xi_3\,,u_{\tau_1}\rangle_{\Gamma_{I_2}} +$$
$$+\langle -\frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{nn}(\varphi)d\xi_3\,,u_{n}\rangle_{\Gamma_{I_3}} +$$
$$+\langle -\frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{n\tau_2}(\varphi)d\xi_3\,,u_{\tau_2}\rangle_{\Gamma_{I_3}} +$$
$$+\langle \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{n\tau_1}(\varphi)d\xi_3\,,u_{\tau_1}\rangle_{\Gamma_{I_3}}.$$

Since the Steklov−Poincare operator for the problem of linear elasticity theory is linear, continuous and coercive on $\left(H^{\frac{1}{2}}(\Gamma_I)\right)^3$ (Hsiao and Wendland, 2008), and using the continuous and compact embedding $H^1(D) \subset\subset H^{\frac{1}{2}}(D)$ for a strong Lipschitz domain D (Hsiao and Wendland, 2008), we get that the operator S_1 is linear, continuous and positive on Λ (assuming that the corresponding tensions are prescribed as boundary conditions on each part of the interface Γ_I).

It is obvious that the Steklov-Poincare operator S is therefore linear, continuous and coercive.

As a result, the preconditioner operators Q, Q_1 and Q_2 are also linear and continuous, and the operators Q and Q_2 are coercive. Moreover, operator Q_2 is symmetric.

By Lax-Milgram lemma, the corresponding Steklov-Poincare equation has unique solution.

Let us state the theorem about the convergence of domain decomposition algorithm (Niemi et al., 2010).

Theorem: (the convergence of domain decomposition algorithm).

Let:
- operator Q_2 be continuous and coercive on a Hilbert space X;
- operator Q_1 be continuous on X;
- operator Q_2 be symmetric and operator Q be coercive on X.

Then for arbitrary $\lambda^0 \in X$ iterations:

$$\lambda^{k+1} = \lambda^k + \theta \mathrm{Q}_2^{-1}\left(\mathrm{G} - \mathrm{Q}\lambda^k\right)$$

converge in X to the solution of the equation:

$$\mathrm{Q}\lambda = \mathrm{G}$$

for arbitrary θ satisfying $0 < \theta < \theta_{max}$.

Therefore, we have formulated and proven the following

Theorem: Let:
- the Steklov-Poincare operator corresponding to the problem of linear elasticity (1) with corresponding boundary conditions be continuous, symmetric and coercive on the corresponding trace spaces defined in Hsiao and Wendland (2008);
- the assumptions of Lemma hold;
- $A_1, A_2, k_1, k_2 \in L^2(\Omega_2^*)$.
- Then the iterative Dirichlet-Neumann scheme for the problem (1)-(4), (6)-(9) with the Dirichlet boundary conditions imposed on the outer edge of the thin part is convergent for some relaxation parameter θ where $0 < \theta < \theta_{max}$.

Proof: Follows from the theorem about the convergence of domain decomposition algorithms (Dirichlet-Neumann scheme) (Quarteroni and Valli, 1999).

4. CONCLUSIONS

We propose a domain decomposition algorithm for the computation of the stress-strain state of the body with thin inclusion. Based on the fact, that the corresponding problems in both parts can be solved separately, one can efficiently solve them preserving the structure and properties of the resulting matrices in both parts. Since the inclusion is modeled using Timoshenko shell theory, the dimension of the problem in the thin part is decreased.

We prove that the corresponding Steklov-Poncare interface equation is well-posed and that the proposed algorithm converges for the appropriately chosen relaxation parameter, which gives the theoretical background for implementation of the proposed algorithm.

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