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THE ANALYSIS OF TRIBOLOGICAL PROCESSES IN THE INKING UNIT OF THE OFFSET PRINTING MACHINE

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Abstract: In this paper is proposed the mathematical description of the temperature distribution resulting from the friction between the two inking rollers (one of which is made off steel and the second one has elastic layer) in the offset printing machine. So-called in printing industry steel vibrator roller perform simultaneously rotary and reciprocating motion. This reciprocating motion is the main source of the heat generation. Using the Laplace transform method for heat conduction equations with boundary conditions taking into account the real processes taking place in the inking unit in contact area we obtained and analyzed the solution that could be useful for determination and regulation of parameters in order to decrease time of process stabilization.

Key words: Printing, Friction, Inking Unit, Laplace Transform, Heat Distribution, Roller/Roller Contact

1. INTRODUCTION

During the printing process in the offset printing machine, temperature of the rollers in the inking unit and surrounding air increases gradually. This phenomena bases on the influence of the pressure between rotating rigid steel and rubber coated rollers and influence of the reciprocating motion along the roller pivot. The phenomena of temperature variations in the inking unit of the printing machines, particularly the offset printing machines which are characterised by a complex inking unit, is inevitable. Even in the machines, in which the inking units (more precisely selected inking rollers) are thermostatted, the fluctuations of the temperature are considerable (Chou et al., 1996). The main cause of temperature increase in the inking unit is the friction between the flexible and rigid axially oscillating rollers. Temperature variations affect the quality of prints. The temperature increase in the inking unit leads to:

- changes of the rheological ink properties (especially viscosity);
- the instability of emulsion of ink and dampening solution;
- the deposition of dust on a rubber blanket;
- increased demand for dampening solution.

This all negatively affects quality of prints. Therefore it seems to be important and reasonable to study this phenomena more carefuly and to built and solve the appriopriate mathematical model. Mathematical models describing the various processes in the mechanical frictional contact presented by Awrejcewicz et al. (2013), Olejnik et al. (2013), Talati et al. (2009), Yevtushenko et al. (2009, 2012) were studied in order to compare various approaches and choose the own path.

2. GEOMETRY AND ASSUMPTIONS OF MODEL

In the inking unit of the offset press the ink is transported from the ink fountain to the plate cylinder through a set of rotating and being in contact rigid and covered with flexible layer rollers. Particular importance for the warm-up phenomenon have steel vibrator rollers which simultaneously perform a rotary (frequency Ω) and reciprocating motion.

Due to the high real speed of rotating rollers (rigid vibrator roller is cooled by the liquid flowing inside the roller), the problem of the roller pair is simplified and reduced to the problem of the roller with liquid coolant inside $T_1^0(t) = \theta_1^0 h_1(t)$ and roller's surrounding, which on the border with the surrounding reaches the temperature ambient (surrounding) temperature $T_2^0(t) = \theta_2^0 h_2(t)$ (Pyryev et al., 2010). The generated heat is a result of friction on the contact surface between rollers. The heat flux acts on the border of the roller $R = R_0$ and surrounding (Fig. 1).

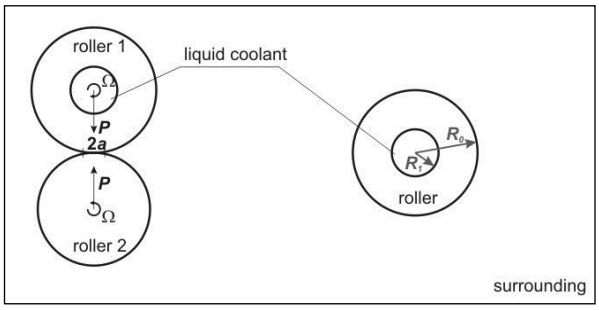

Fig. 1. The transition from the real issue to the model

In the case of axisymmetric state of roller temperature T_1 state the heat conduction equation for an isotropic body in cooperation with cylindrical coordinates (R, φ, z) given by Carslaw et. al (1959) will take the form as follows:

$$\frac{\partial^2 T_1(R,t)}{\partial R^2} + \frac{1}{R}\frac{\partial T_1(R,t)}{\partial R} = \frac{1}{k_1}\frac{\partial T_1(R,t)}{\partial t}, \quad R_1 < R < R_0 \qquad (1)$$

with following boundary conditions (after Awrejcewicz et al., 2009):

$$\lambda_1 \frac{\partial T_1(R_1,t)}{\partial R} - \alpha_1^T [T_1(R_1,t) - \theta_1^0 h_1(t)] = 0 \tag{2}$$

$$\lambda_1 \frac{\partial T_1(R_0,t)}{\partial R} + \alpha_2^T [T_1(R_1,t) - \theta_2^0 h_2(t)] = Q(t) \tag{3}$$

and initial conditions:

$$T_1(R,0) = 0\,,\ R_1 < R < R_0 \tag{4}$$

Above we have taken indications as follows: $T_1(R,t)$– temperature increase of the roller, t – time, $Q(t) = \theta_0^0 h_0(t)$ – power density of friction, k_1 – thermal diffusivity of a roller, α_n^T – heat transfer coefficient: $(n = 1)$ between the coolant and material which the roller is made of (forced water flow in the pipe), $(n = 2)$ between the surrounding (air) and material which the roller is made of (forced air flow near the surface of the roller), λ_1 – thermal conductivity of the roller, $h_n(t)$ – dimensionless internal $(n = 1)$ and external $(n = 2)$ temperature, $T_1^0(t) = \theta_1^0 h_1(t)$, $T_2^0(t) = \theta_2^0 h_2(t)$ – temperature increase of the coolant and sorrounding.

We assume that the reciprocating movement (and resulted from it friction) is the source of heat, and hence we assumed the formula of the friction power density:

$$Q(t) = (1-\eta_1)\eta_2\, f(V_r) V_r\ p_0 \tag{5}$$

where: p_0 – average contact pressure, associated with a contact force P by formula $p_0 = P/2a$, where contact area width $2a$ is found from Herz problem solution for two rollers (Jurkiewicz et al., 2011), V_r – relative velocity of being in contact rollers, $f(V_r)$ – kinematic friction coefficient, η_1 – coefficient responsible for the part of the power which is lost e.g. because of wear, η_2 – coefficient depending on time.

After Blok (1940) we assume that kinematic friction coefficient: $f(V_r) = f_0 sgn(V_r)$ and therefore we obtain:

$$Q(t) = (1-\eta_1)\eta_2\ f_0 |V_r|\, p_0 \tag{6}$$

The construction and operation of the inking unit (rotary and reciprocating motion of the roller) results that the relative movement of being in contact rollers along the axis changes periodically:

$$Z(t) = Z_0 \sin \omega t \tag{7}$$

where: – maximum relative displacement of the rollers, ω – frequency of relative displacement of rollers (from the construction of the inking unit $\omega = \Omega$).

The fact that the heat is generated by friction in changing contact area:

$$S(t) = 2a\left(L - |Z(t)|\right) \tag{8}$$

where: $S(t)$ – contact area between rollers, L – length of the roller,is included by entering the depending on the time coefficient η_2 into the formula (6) of the heat flux $Q(t)$:

$$\eta_2 = S(t)/S_1 \tag{9}$$

where: S_1 – area of the cylinder side $S_1 = 2\pi R_0 L$.

Hence the coefficient:

$$\eta_2 = \frac{2a\left(L - |Z(t)|\right)}{2\pi R_0 L} = \frac{a}{\pi R_0}\left(1 - \frac{Z_0}{L}|\sin \omega t|\right) \tag{10}$$

The heat flux can now be written as:

$$Q(t) = (1-\eta_1)\frac{a f_0 \omega Z_0 p_0}{\pi R_0}\left(1 - \frac{Z_0}{L}|\sin \omega t|\right)|\cos \omega t| \tag{11}$$

Assuming:

$$Q(t) = \theta_0^0 h_0(t) \text{ i } z_0 = \frac{Z_0}{L} \tag{12}$$

we obtain:

$$\theta_0^0 = (1-\eta_1)\frac{a f_0 \omega Z_0 p_0}{\pi R_0} \tag{13}$$

$$h_0(t) = \left(1 - z_0 |\sin \omega t|\right)|\cos \omega t| \tag{14}$$

Below, in Fig. 2, there is shown an exemplary graph of the function $h_0(t)$ for $z_0 = 0.2$.

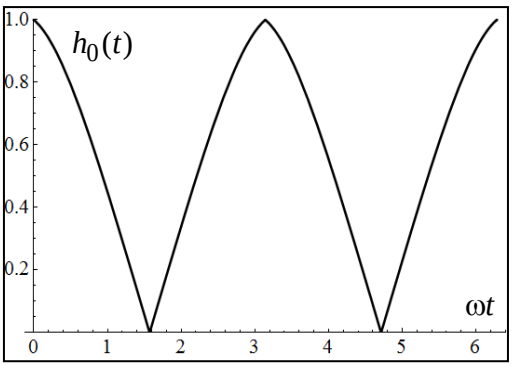

Fig. 2. The evolution of the heat flux due to the rollers' reciprocation motion

3. NON-STATIONERY SOLUTION

On the proposed model equation (1) we perform Laplace transform (Carslaw et al., 1959; Abramowitz et al., 1965) defined by the relationship (s – transform parameter):

$$\left\{\overline{T}_1(R,s), \overline{h}_n(s), \overline{Q}(s)\right\} = \int_0^\infty \left\{T_1(R,t), h_n(t), Q(t)\right\} e^{-st} dt \tag{15}$$

As a result of the Laplace transform we obtain second-order differential equation (*modified Bessel equations*):

$$\frac{\partial^2 \overline{T}_1(R,s)}{\partial R^2} + \frac{1}{R}\frac{\partial \overline{T}_1(R,s)}{\partial R} = \frac{s}{k_1}\overline{T}_1(R,s)\,,\ R_1 < R < R_0 \tag{16}$$

and boundary conditions:

$$\lambda_1 \frac{\partial \overline{T}_1(R_1,s)}{\partial R} - \alpha_1^T \left[\overline{T}_1(R_1,s) - \overline{T}_1^0(s)\right] = 0 \tag{17}$$

$$\lambda_1 \frac{\partial \overline{T}_1(R_0,s)}{\partial R} + \alpha_2^T \left[\overline{T}_1(R_0,s) - \overline{T}_2^0(s)\right] = \overline{Q}(s) \tag{18}$$

The general solution of the equations system (16)-(18) has the form (Carslaw et al., 1959; Abramowitz et al., 1965):

$$\overline{T}_1(R_1,s) = C_3^1 I_0(q_1 R) + C_4^1 K_0(q_1 R),\ q_1 = \sqrt{\frac{s}{k_1}} \tag{19}$$

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We find required differential (Abramowitz et al., 1965):

$$\frac{d\bar{T}_1(R,s)}{dR} = q_1 C_3^1 I_1(q_1 R) - q_1 C_4^1 K_1(q_1 R) \tag{20}$$

After introducing equations (19)-(20) into the boundary conditions (17)-(18) we get a set of two equations with two unknowns: C_1^1 and C_2^1:

$$\lambda_1\left(s_1 C_3^1 I_1(s_1 R_1) - s_1 C_4^1 K_1(s_1 R_1)\right) - \alpha_1^T\left(C_3^1 I_0(s_1 R_1) + C_4^1 K_0(s_1 R_1)\right) = -\alpha_1^T \theta_1^0 \bar{h}_1(s) \tag{21}$$

$$\lambda_1\left(s_1 C_3^1 I_1(s_1 R_0) - s_1 C_4^1 K_1(s_1 R_0)\right) + \alpha_2^T\left(C_3^1 I_0(s_1 R_0) + C_4^1 K_0(s_1 R_0)\right) = +\alpha_2^T \theta_2^0 \bar{h}_2(s) + \bar{Q}(s) \tag{22}$$

By entering constants obtained from the equations (21)-(22) into the general solution (19) we receive the final solution and write it in form:

$$\bar{T}_1(R,s) = \sum_{n=0}^{2}\left[\frac{\bar{\theta}_{1n}(R,s)}{\Delta(s)}\theta_n^0 \bar{h}_n(s)\right] \tag{23}$$

where: $\Delta(s) = Bi_2 Q_2^{(1)}(R_0,R_1,s) + Q_1^{(1)}(R_0,R_1,s)$

$\bar{\theta}_{10}(R,s) = R_0\lambda_1^{-1}Q_2^{(1)}(R,R_1,s) = (\alpha_2^T)^{-1}Bi_2 Q_2^{(1)}(R,R_1,s)$

$\bar{\theta}_{11}(R,s) = Bi_1 Q_1^{(2)}(R_0,R,s)$, $\bar{\theta}_{12}(R,s) = Bi_2 Q_2^{(1)}(R,R_1,s)$

$Q_2^{(1)}(x,y,s) = Bi_1 Z_{00}^{(1)}(x,y,s) + Z_{10}^{(1)}(y,x,s)$

$Q_1^{(2)}(x,y,s) = Bi_2 Z_{00}^{(1)}(x,y,s) + Z_{10}^{(1)}(x,y,s)$

$Z_{00}^{(1)}(x,y,s) = I_0(q_1 x)K_0(q_1 y) - I_0(q_1 y)K_0(q_1 x)$

$Z_{11}^{(1)}(x,y,s) = I_1(q_1 x)K_1(q_1 y) - I_1(q_1 y)K_1(q_1 x)$

$Z_{10}^{(1)}(x,y,s) = q_1 x\left(I_1(q_1 x)K_0(q_1 y) + I_0(q_1 y)K_1(q_1 x)\right)$

By entering obtained from set of equations (21)-(22) constants into general solution (20) we receive the form of heat flux:

$$\bar{Q}_1(R,s) = \lambda_1\frac{\partial\bar{T}_1(R,s)}{\partial R} = \sum_{n=0}^{2}\left[\frac{\bar{Q}_{1n}(R,s)}{\Delta(s)}\theta_n^0 \bar{h}_n(s)\right] \tag{24}$$

where:

$\bar{Q}_{10}(R,s) = \lambda_1 R_0\left(Bi_1 Z_{10}^{(1)}(R,R_1,s)/R - q_1 q_1 R_1 Z_{11}^{(1)}(R_1,R,s)\right)$

$\bar{Q}_{11}(R,s) = \lambda_1 Bi_1 Q_1^{(2)}(R_0,R,s) = \lambda_1 Bi_1\left(-Bi_2 Z_{10}^{(1)}(R,R_0,s)/R - q_1 q_1 R_0 Z_{11}^{(1)}(R_0,R,s)\right)$

$\bar{Q}_{12}(R,s) = \lambda_1 Bi_2 Q_2^{(1)}(R,R_1,s) = \lambda_1 Bi_2\left(Bi_1 Z_{10}^{(1)}(R,R_1,s)/R - q_1 q_1 R_1 Z_{11}^{(1)}(R_1,R,s)\right)$

$Q_2^{(1)}(x,y,s) = Bi_1 Z_{00}^{(1)}(x,y,s) + Z_{10}^{(1)}(y,x,s)$

$Q_1^{(2)}(x,y,s) = Bi_2 Z_{00}^{(1)}(x,y,s) + Z_{10}^{(1)}(x,y,s)$

Above we have taken indications as follows: $I_m(z)$, $K_m(z)$ – modified mth-order Bessel functions, of first and second kind respectively Abramowitz et al., 1965), $B_{i1} = R_1\alpha_1^T/\lambda_1$, $B_{i2} = R_0\alpha_2^T/\lambda_1$ – Biot's numbers (Carslaw et al., 1959). Temperature transform $\bar{T}_1(R,s)$ and heat flux transform have been written in forms (23) and (24) by means of functions $Z_{ml}^{(n)}(x,y,s)$, which are defined above.

The Laplace transform of periodic function (14) has the form:

$$\bar{h}_0(s) = \frac{2\omega e^{\pi s/2\omega} - s + se^{\pi s/\omega}}{(e^{\pi s/\omega}-1)(s^2+\omega^2)} - \frac{3z_0(e^{\pi s/2\omega}+1)}{(e^{\pi s/2\omega}-1)(s^2+4\omega^2)} \tag{25}$$

4. ANALYSIS

The analysis of obtained Laplace transforms shows that the peculiarities of these transforms can be only zero points of the characteristic function $\Delta(s)$ and the peculiarities of Laplace transforms $\bar{h}_n(s)$.

Using the Laplace transform method and Borel convolution theorem we find the solution and write roller temperature and the heat flux in the forms as follows (Piętak et al., 2011; Jurkiewicz et. al., 2011):

$$T_1(R,t) = \sum_{n=0}^{2}\left[\frac{\bar{\theta}_{1n}(R,0)}{\Delta(0)} + \sum_{k=1}^{\infty}\frac{\bar{\theta}_{1n}(R,s_k)}{s_k\Delta'(s_k)}\exp(s_k t)\right] * \frac{d}{dt}\left(\theta_n^0 h_n(t)\right) \tag{26}$$

$$Q_1(R,t) = \sum_{n=0}^{2}\left[\frac{\bar{Q}_{1n}(R,0)}{\Delta(0)} + \sum_{k=1}^{\infty}\frac{\bar{Q}_{1n}(R,s_k)}{s_k\Delta'(s_k)}\exp(s_k t)\right] * \frac{d}{dt}\left(\theta_n^0 h_n(t)\right) \tag{27}$$

where: $\Delta'(s_k) = d\Delta(s)/ds|_{s=s_k}$.

Roots s_k of the characteristic equation $\Delta(s) = 0$ lie on the negative part of the real axis of complex plane of Laplace transform parameter, i.e.: $s_k = -\mu_k^2$.

For example, the thermostatted vibrator roller in the machine Heidelberg XL105 (according to technical drawing) has a diameter $2R_0 = 84.8\cdot10^{-3}\text{m}$ and length $L = 1085\cdot10^{-3}\text{m}$, width of the contact zone $2a = 5\cdot10^{-3}\text{m}$, displacement of the roller along the axis $z_0 = 17.5\cdot10^{-3}\text{m}$, coolant flow diameter $2R_1 = 15\cdot10^{-3}\text{m}$. We assume that:

- the roller is made of steel (according to VanSant (1983): $\lambda_1 = 21\ W/(m°\text{C})$, $k_1 = 0.7\cdot10^{-5}\text{m}^2/s$);
- the cooling fluid is the water at a temperature $\theta_1^0 h_1(t)$ under conditions of forced flow into the pipe ($\alpha_1^T = 2000W/(m°\text{C})$);
- and the surrounding is the air at the temperature $\theta_2^0 h_2(t)$ under conditions of forced flow near the surface of the roller ($\alpha_2^T = 10W/(m°\text{C})$).

The preliminary analysis was performed for the case when functions $h_n(t)$ for $n = 0, 1, 2$ are Heaviside step functions $H(t)$ ($H(t) = 0$ for $t < 0$ and $H(t) = 1$ for $t < 0$). Instead of periodic function $h_0(t)$ with the period π/ω we took in to account also the Heviside function $H(t)$ with the amplitude equel to the average value h_0 of this function:

$$h_0 = \frac{\omega}{\pi}\int_0^{\pi/\omega} h_0(t)dt = \frac{2-z_0}{\pi} \tag{28}$$

We examine the case where at the surface of the roller $R = R_0$ due to the friction acts the heat flux Q with amplitude $\theta_0^0 = 4000\text{W/m}^2$, the surrounding temperature increases in amount of $\theta_2^0 = 5°\text{C}$, and the coolant temperature is lower in amount of $\theta_1^0 = -20°\text{C}$.

First roots of the characteristic equation $\Delta(s) = 0$ are as follows: $s_1 = -3.39 \cdot 10^{-3}$, $s_2 = -8.55 \cdot 10^{-2}$, $s_3 = -2.68 \cdot 10^{-1}$, $s_4 = -5.57 \cdot 10^{-1}$, $s_5 = -9.57 \cdot 10^{-1}$, $s_6 = -1.47$, $s_7 = -2.09$, $s_8 = -3.83$.

Taking into account the the first 17 roots of the chareacteristic equation in the formula (26) we show in Fig. 3 the evolution of the changes in the temperature of the roller surface $R = R_0$ (curve 1).

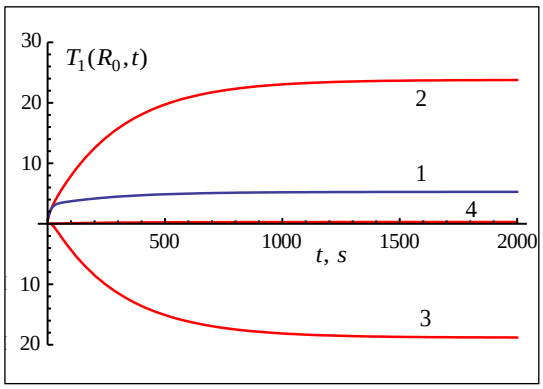

Fig. 3. The evolution of temperature changes at the roller surface $R = R_0$

Curve 2 presents roller surface temperature changes in case when the surrounding temperature increase and the coolant temperature change are both zero. The change of the roller surface temperature ($R = R_0$) in time in case when there is no friction and no changes of the surrounding temperature shows the curve 3 in Fig. 3. The change in roller surface temperature ($R = R_0$) caused by the increase of the surrounding temperature illustrates the curve 4 in Fig. 3.

The evolution of the roller temperature at surface $R = R_1$ we present in Fig. 4 (curve 1). Curve 2 in Fig. 4 describes evolution of roller internal surface ($R = R_1$) temperature which is a result of the heat flux operating at the surface $R = R_0$ in case when the surrounding temperature is constant ($\theta_1^0 h_1(t) = 0$) and there is no change in the coolant temperature. Changes in the temperature at internal roller surface in case when there is no friction and no changes in surrounding temperature are presented by curve 3 in Fig. 4. Changes in the temperature at internal surface $R = R_1$ caused by increase in the surrounding temperature shows curve 4 in Fig. 4.

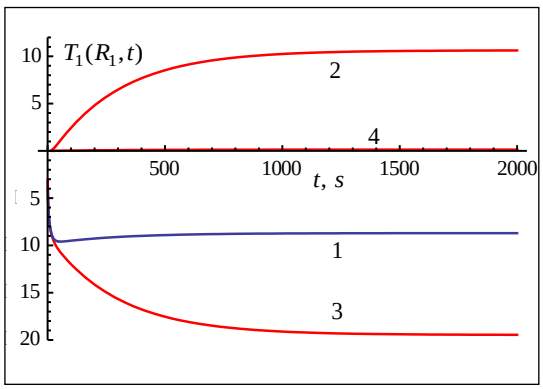

Fig. 4. The evolution of temperature changes at the roller surface $R = R_1$

The roller temperature distribution $T_1(R,t)$ along the roller radius R is shown in Fig. 5. Curves 1-5 correspond to the temperature distribution, respectively at time $t = 10,\ 50, 100,\ 500, 2000$ s.

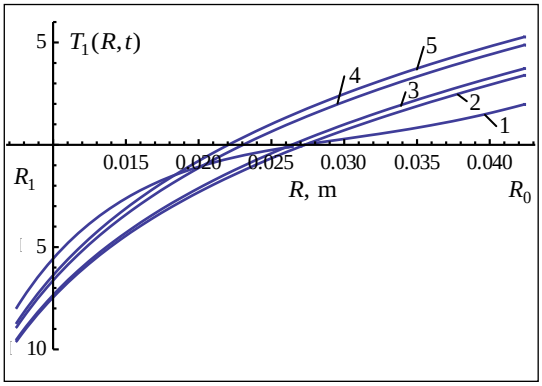

Fig. 5. The temperature distribution along the roller radius at different time instants

5. CONCLUSIONS

We obtain an analytical solution of the heat conduction problem for the inking unit in the offset printing machine. It has been proven that the functions $Z_{ml}^{(n)}(x,y,s)$, which we entered into transforms' formulas of the roller temperature (23) and the heat flux (24), are analytic functions.

It is worth noting, that the first terms of obtained formulas of roller temperature (26) and heat flux (27) corresponds to the stationary solution ($t \to \infty$ of this problem received in the earlier work (Piętak et al., 2011) for the case when the functions $h_n(t)$ where $n = 0, 1, 2$ are Heaviside step functions (unit step function).

We assume that the inking is at temperature $22°C$. During the printing process (when machine is working) the heat flux is generated at the surface of the vibrator roller ($R = R_0$) as a result of friction. The heat flux causes heating of the roller. If there is no cooling, the temperare increase overe time in amount of $24°C$ and reaches the level of $46°C$ (curve 2, Fig. 3). Offset printing technology requires lower temperature of the roller surface and of the ink transported by the roller. Therfore, in modern inking units rollers are thermostatted. When we take into account the coolant inside the roller we receive an increase of roller surface temperature in amount of only $5°C$ (curve 1 in Fig. 3). This way our model allows to simulate the temperature distribution in one of the elements of the prinitng unit of the offset printing machine and to regulate coolant temperature depending on the surrounding temperature changes and printing parameters, such as speed and properties of the ink.

The obtained solution makes it possible to determine and to regulate the time necessary to reach the stationery coditions of the temperature. Stationery conditions of the temeperature gurantee the stability of the printing process and less paper waste. In this way we can affect the amount of foracasted paper and ink waste.

We woluld like also notice that various forms of solutions for the cylinder (roller) are showed, e.g. in works of Carslaw et al. (1959) and VanSant (1983).

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