DOI 10.2478/ama-2018-0041

TWO-PARAMETRIC ANALYSIS OF ANTI-PLANE SHEAR DEFORMATION OF A COATED ELASTIC HALF-SPACE

Leyla SULTANOVA*

*School of Computing and Mathematics, Keele University, Keele, Staffordshire, ST5 5BG, United Kingdom

l.sultanova@keele.ac.uk

received 31 October 2018, revised 6 December 2018, accepted 10 December 2018

Abstract: The anti-plane shear deformation problem of a half-space coated by a soft or a stiff thin layer is considered. The two-term asymptotic analysis is developed motivated by the scaling for the displacement and stress components obtained from the exact solution of a model problem for a shear harmonic load. It is shown that for a rather high contrast in stiffness of the layer and the half-space Winkler-type behaviour appears for a relatively soft coating, while for a relatively stiff one, the equations of plate shear are valid. For low contrast, an alternative approximation is suggested based on the reduced continuity conditions and the fact that the applied load may be transmitted to the interface. In case of a stiff layer, a simpler problem for a homogeneous half-space with effective boundary condition is also formulated, modelling the effect of the coating, while for a relatively soft layer a uniformly valid approximate formula is introduced.

Key words: Soft/Stiff Thin Coating, Asymptotic, Contrast, Substrate

1. INTRODUCTION

Coated structures find numerous applications in modern engineering and technology, including, in particular biological sciences and structural mechanics, see e.g. Li et al. (2014), Bose (2017) and Pawlowski (2008). The presence of a thin coating layer usually motivates an asymptotic approach relying on a small geometric parameter, see e.g. Ahmad et al. (2011), Kaplunov and Prikazchikov (2017), and Yang (2006). Often, in addition, there is a contrast in material parameters of the coating and the half-space, hence, the problem could require multiparametric analysis, similar to that presented recently by Kaplunov et al. (2016) and Kaplunov et al. (2017) for vibrations of strongly inhomogeneous structures. In case of a coated half-space, high contrast in stiffness between the layer and the substrate implies a second small material parameter, along with the two limiting cases, corresponding to a relatively soft and a relatively stiff coating. The importance of these two cases was appreciated within the framework of contact problems, see e.g. Alexandrov (2010). A two-parametric asymptotic analysis of equilibrium of a 3D half-space coated by a soft layer, allowing a variety of scenarios depending on the relation between the relative thickness and stiffness, has been carried out by Kaplunov et al. (2018).

In this paper, these results are extended to a problem of anti-plane shear deformation of a coated elastic half-space. Focusing on an anti-plane shear is of interest within linear and nonlinear solid mechanics theories, since it allows establishing a mathematically simpler analysis without loss of physical interpretation, see e.g. Horgan (1995). First, we derive the exact solution for anti-plane deformations caused by a harmonic shear load. Then, considering the relative thickness of the layer to be small, and supposing a contrast in stiffness of the layer and the half-space, we develop a two-parametric asymptotic analysis for an arbitrary shear load. While doing it, we rely on the exact solution in order to motivate the original scaling of the displacement and stresses, required for the asymptotic integration technique, see for more detail Aghalovyan (2015), Argatov and Mishuris (2016), and Goldenveizer et al. (1993). A classification following from the relation between the two asymptotic parameters is established. The results for the anti-plane displacement and stress components are obtained. In particular, we focus on the relation between the shear load and the displacement, which results in Winkler-type behaviour for a rather soft coating and involves the equations of plate shear in case of a soft layer. The derived asymptotic results are compared numerically with the exact solution for shear harmonic load.

2. STATEMENT OF THE PROBLEM

Consider an anti-plane problem of equilibrium for a homogeneous linearly elastic isotropic half-space coated by a thin isotropic layer of thickness h, subject to action of a shear force $P = P(x_1)$ at the surface of the coating $(x_3 = 0)$, see Fig. 1.

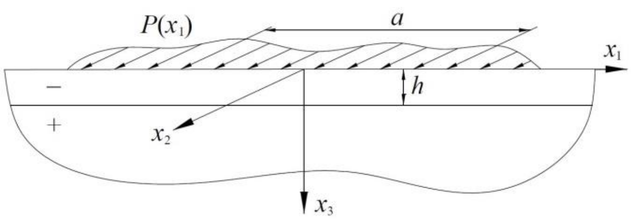

Fig. 1. Problem statement

Throughout the paper, we assume the following:
– the thickness of the layer h is small compared to a typical length scale a related to the load variation along the coordinate x_1;

– there is a contrast in stiffness of the layer and the half-space;

In view of these assumptions, we introduce a small geometrical parameter:

$$\varepsilon = \frac{h}{a} \ll 1, \tag{1}$$

and a material parameter:

$$\mu = \begin{cases} \frac{\mu^-}{\mu^+}, & \mu^- \le \mu^+ \\ \frac{\mu^+}{\mu^-}, & \mu^+ \le \mu^- \end{cases} \lesssim 1, \tag{2}$$

where: $\mu^\pm$ – shear moduli, with "$-$" and "$+$" denoting the layer and the half-space, respectively. The first line in (2) corresponds to the case of the soft layer, and the second line is for a relatively stiff coating. Note, that the non-nontrast case ($\mu = 1$) is also included in consideration. The parameters above may be related to each other as:

$$\mu = \varepsilon^\alpha, \quad \alpha \ge 0, \tag{3}$$

where for a fixed ε, α represents the level of the contrast, i.e. with an increase of α, the contrast in stiffness of the layer and the half-space becomes more pronounced.

In this paper, we concentrate on the anti-plane problem assuming, therefore, that displacements $u_1^\pm = u_3^\pm = 0$ and $u_2^\pm$ do not depend on x_2. Hence, defining dimensionless variables as:

$$\xi_1 = \frac{x_1}{a}, \tag{4}$$

$$\xi_3^- = \frac{x_3}{h}, \quad 0 \le x_3 \le h, \quad \xi_3^+ = \frac{x_3 - h}{a}, \quad x_3 \ge h, \tag{5}$$

the governing equations for the layer and the half-space are written as:

$$\frac{h}{a}\sigma_{12,1}^- + \sigma_{23,3}^- = 0, \quad \sigma_{12}^- = \frac{\mu^-}{a}u_{2,1}^-, \quad \sigma_{23}^- = \frac{\mu^-}{h}u_{2,3}^-, \tag{6}$$

$$\sigma_{12,1}^+ + \sigma_{23,3}^+ = 0, \quad \sigma_{12}^+ = \frac{\mu^+}{a}u_{2,1}^+, \quad \sigma_{23}^+ = \frac{\mu^+}{a}u_{2,3}^+, \tag{7}$$

where: $\sigma_{12}^\pm$ and $\sigma_{23}^\pm$ – Cauchy stresses, and comma indicates differentiation.

The boundary condition, modelling shear load at the surface of the layer, and continuity conditions at the interface take the form

$$\sigma_{23}^- = -P, \quad \xi_3^- = 0, \tag{8}$$

$$u_2^- = u_2^+, \quad \sigma_{23}^- = \sigma_{23}^+, \quad \xi_3^- = 1. \tag{9}$$

We also impose the decay condition for the displacement, i.e. $u_2^+ \to 0$ as $\xi_3^+ \to \infty$.

3. PROBLEM FOR A HARMONIC SHEAR LOAD

We begin the analysis with investigation of a model problem for a shear harmonic force

$$P = A\mu^- \sin\xi_1, \tag{10}$$

where: A – constant amplitude, see Fig. 2.

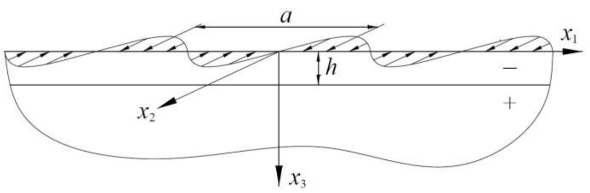

Fig. 2. Problem for a harmonic surface load

In this case, the displacemens may be sought as:

$$u_2^\pm = f^\pm(\xi_3^\pm)\sin\xi_1. \tag{11}$$

Substituting (11) into governing equations (6) and (7), we have:

$$f^{-''}(\xi_3^-) - \varepsilon^2 f^-(\xi_3^-) = 0, \quad f^{+''}(\xi_3^+) - f^+(\xi_3^+) = 0. \tag{12}$$

Taking into account boundary and continuity conditions (8) and (9), respectively, the solution of (12) decaying at infinity is written as:

$$f^-(\xi_3^-) = c_1 e^{\varepsilon\xi_3^-} + c_2 e^{-\varepsilon\xi_3^-}, \quad f^+(\xi_3^+) = c_3 e^{-\xi_3^+}, \tag{13}$$

where:

$$c_i = \frac{N_i}{D}, \quad i = 1,2,3, \tag{14}$$

with:

$$N_1 = Ah(\mu^- - \mu^+), \quad N_2 = Ahe^{2\varepsilon}(\mu^- + \mu^+), \tag{15}$$

$$N_3 = 2Ahe^\varepsilon\mu^-, \tag{16}$$

and:

$$D = \varepsilon[\mu^-(e^{2\varepsilon} - 1) + \mu^+(e^{2\varepsilon} + 1)]. \tag{17}$$

Substituting the latter into relations (6) and (7), the stress components are found in the form:

$$\sigma_{12}^\pm = \frac{\mu^\pm}{a} f^\pm(\xi_3^\pm)\cos\xi_1, \quad \sigma_{23}^+ = -\frac{\mu^+}{a} f^+(\xi_3^+)\sin\xi_1, \tag{18}$$

$$\sigma_{23}^- = \frac{\mu^-}{h}\varepsilon\left(c_1 e^{\varepsilon\xi_3^-} - c_2 e^{-\varepsilon\xi_3^-}\right)\sin\xi_1. \tag{19}$$

Then, using (1) and (2), together with (3), we deduce the leading order asymptotic behaviour of the displacement and stress components obtained above, in terms of a small parameter ε, for a relatively soft and stiff layer, see Table 1.

Tab. 1. Asymptotic behaviour of displacements and stresses

	Soft layer		Stiff layer	
	$\alpha \ge 1$	$0 \le \alpha \le 1$	$\alpha \ge 1$	$0 \le \alpha \le 1$
u_2^-	1	$\varepsilon^{\alpha-1}$	ε^{-2}	$\varepsilon^{-\alpha-1}$
σ_{12}^-	ε	ε^α	ε^{-1}	$\varepsilon^{-\alpha}$
σ_{23}^-	1	1	1	1
u_2^+	$\varepsilon^{\alpha-1}$	$\varepsilon^{\alpha-1}$	ε^{-2}	$\varepsilon^{-\alpha-1}$
σ_{12}^+	1	1	$\varepsilon^{\alpha-1}$	1
σ_{23}^+	1	1	$\varepsilon^{\alpha-1}$	1

Now, we study in more detail the relation between displacement u_2^- at the surface of the coating ($\xi_3^- = 0$) and prescribed load P by introducing the coefficient:

$$k = \frac{P}{u_2^-|_{\xi_3^-=0}}. \tag{20}$$

Note, that this coefficient is constant only for the considered sinusoidal load. In general, the relation between displacement $u_2^-|_{\xi_3^-=0}$ and load P is a function of ξ_1. In case of a shear harmonic load (10), it is given by:

$$k = \frac{A\mu^- D}{N_1 + N_2}, \tag{21}$$

following from (11), $(13)_1$ and (14). The leading order estimates of the coefficient k depending on the parameter α are presented in Table 2.

DOI 10.2478/ama-2018-0041

Tab. 2. Leading order of the coefficient k for a harmonic shear force

	k	
	Soft layer	**Stiff layer**
$\alpha > 1$	$\frac{\mu^-}{h}$	$\frac{\mu^- h}{a^2}$
$\alpha = 1$	$\frac{\mu^-\mu^+}{h\mu^+ + a\mu^-}$	$\frac{2\mu^- h}{a^2}$
$0 \le \alpha < 1$	$\frac{\mu^+}{a}$	$\frac{\mu^+}{a}$

Therefore, at $\alpha > 1$, the coefficient k does not depend on the material parameter of the half-space μ^+, meaning that the deformation of the substrate is neglected. In general, in case of a rather soft layer, it may be described as a Winkler-type behaviour, similarily to Kaplunov et al. (2018), while for a stiff layer, taking into account the term a^2, it indicates that the plate shear equation may be expected as a relation between u_2^- and P. In the range $0 \le \alpha < 1$, the relation is entirely affected by the presence of the half-space, i.e. the layer may no longer be separated, and the original problem for a coated solid should be considered. The case $\alpha = 1$ seems to be a transition point, since, for instance, for a soft layer, k depends on both the material parameters of the layer and the half-space, but, at the same time, according to assumptions (1), (2) and the relation (3), $\frac{h}{a} = \frac{\mu^-}{\mu^+} = \varepsilon$, therefore, it may also be written as $k = \frac{\mu^-}{2h}$.

4. ASYMPTOTIC ANALYSIS

In this section we develop a more general procedure for an arbitrary load acting on the surface of the layer, adopting the method of direct asymptotic integration of the equations of elasticity. Note that the scaling is motivated by the asymptotic orders in Table 1, obtained for a harmonic shear force.

4.1. Soft layer, $\alpha \ge 1$

First, we scale the displacement and stress components according to the first column of the Table 1, having for a relatively soft layer:

$$u_2^- = h u_2^{*-}, \quad \sigma_{12}^- = \mu^- \varepsilon \sigma_{12}^{*-}, \quad \sigma_{23}^- = \mu^- \sigma_{23}^{*-}, \tag{22}$$

where the quantities with the asterisk are assumed to be of the same asymptotic order. Hence, governing equations (6) become:

$$\varepsilon^2 \sigma_{12,1}^{*-} + \sigma_{23,3}^{*-} = 0, \quad \sigma_{12}^{*-} = u_{2,1}^{*-}, \quad \sigma_{23}^{*-} = u_{2,3}^{*-}. \tag{23}$$

Similarly, substituting the scaling for the half-space given by:

$$u_2^+ = h\varepsilon^{\alpha-1} u_2^{*+}, \quad \sigma_{12}^+ = \mu^- \sigma_{12}^{*+}, \quad \sigma_{23}^+ = \mu^- \sigma_{23}^{*+}, \tag{24}$$

into equations (7), we get:

$$\sigma_{12,1}^{*+} + \sigma_{23,3}^{*+} = 0, \quad \sigma_{12}^{*+} = u_{2,1}^{*+}, \quad \sigma_{23}^{*+} = u_{2,3}^{*+}. \tag{25}$$

Here and below, the applied load is scaled as:

$$P = \mu^- p^*. \tag{26}$$

In what follows, boundary and continuity conditions (8) and (9), respectively, may be rewritten as:

$$\sigma_{23}^{*-} = -p^*, \quad \xi_3^- = 0, \tag{27}$$

$$u_2^{*-} = \varepsilon^{\alpha-1} u_2^{*+}, \quad \sigma_{23}^{*-} = \sigma_{23}^{*+}, \quad \xi_3^- = 1. \tag{28}$$

Next, we expand the displacements and stresses of the layer in asymptotic series:

$$\begin{pmatrix} u_2^{*-} \\ \sigma_{12}^{*-} \\ \sigma_{23}^{*-} \end{pmatrix} = \begin{pmatrix} u_2^{-(0)} \\ \sigma_{12}^{-(0)} \\ \sigma_{23}^{-(0)} \end{pmatrix} + \cdots . \tag{29}$$

Hence, at leading order we have from (23):

$$\sigma_{23,3}^{-(0)} = 0, \quad \sigma_{12}^{-(0)} = u_{2,1}^{-(0)}, \quad \sigma_{23}^{-(0)} = u_{2,3}^{-(0)}, \tag{30}$$

subject to boundary conditions at $\xi_3^- = 0$

$$\sigma_{23}^{-(0)} = -p^*. \tag{31}$$

In view of $(28)_1$, $u_2^{*-} \gg u_2^{*+}$ at $\alpha > 1$ while $u_2^{*-} \sim u_2^{*+}$ at $\alpha = 1$, therefore, the leading order continuity conditions at $\xi_3^- = 1$ become:

$$u_2^{-(0)} = 0, \quad \alpha > 1, \quad u_2^{-(0)} = u_2^{+(0)}, \quad \alpha = 1, \tag{32}$$

$$\sigma_{23}^{-(0)} = \sigma_{23}^{+(0)}. \tag{33}$$

From $(30)_1$ and satisfying (31), we obtain:

$$\sigma_{23}^{-(0)} = -p^*. \tag{34}$$

Then, using $(30)_3$ together with (32), we deduce:

$$u_2^{-(0)} = p^*(1 - \xi_3^-), \quad \alpha > 1, \tag{35}$$

$$u_2^{-(0)} = p^*(1 - \xi_3^-) + u_2^{+(0)}\big|_{\xi_3^- = 1}, \quad \alpha = 1. \tag{36}$$

Therefore, as it was discussed above, at $\xi_3^- = 0$, the relation between displacement and applied load at $\alpha > 1$ is not affected by the presence of the substrate, which may be described as Winkler-type behaviour, while for $\alpha = 1$ the reaction of the half-space is involved. The same happens for shear stress $\sigma_{12}^{-(0)}$, for which we have from $(30)_2$, (35) and (36):

$$\sigma_{12}^{-(0)} = \frac{\partial p^*}{\partial \xi_1}(1 - \xi_3^-), \quad \alpha > 1, \tag{37}$$

$$\sigma_{12}^{-(0)} = \frac{\partial p^*}{\partial \xi_1}(1 - \xi_3^-) + \frac{\partial u_2^{+(0)}}{\partial \xi_1}\bigg|_{\xi_3^- = 1}, \quad \alpha = 1. \tag{38}$$

4.2. Soft layer, $0 \le \alpha < 1$

Scaling for the layer now takes the form:

$$u_2^- = h\varepsilon^{\alpha-1} u_2^{*-}, \quad \sigma_{12}^- = \mu^- \varepsilon^{\alpha} \sigma_{12}^{*-}, \quad \sigma_{23}^- = \mu^- \sigma_{23}^{*-}, \tag{39}$$

leading to:

$$\varepsilon^{\alpha+1} \sigma_{12,1}^{*-} + \sigma_{23,3}^{*-} = 0, \tag{40}$$

$$\sigma_{12}^{*-} = u_{2,1}^{*-}, \quad \varepsilon^{1-\alpha} \sigma_{23}^{*-} = u_{2,3}^{*-}. \tag{41}$$

Scaling and equations for the half-space are taken as (24) and (25), respectively, with boundary condition (27), whereas the continuity conditions become:

$$u_2^{*-} = u_2^{*+}, \quad \sigma_{23}^{*-} = \sigma_{23}^{*+}, \quad \xi_3^- = 1. \tag{42}$$

At leading order, the equations for the layer are:

$$\sigma_{23,3}^{-(0)}=0,\quad \sigma_{12}^{-(0)}=u_{2,1}^{-(0)},\quad u_{2,3}^{-(0)}=0, \tag{43}$$

subject to boundary condition (31) and the following continuity conditions:

$$u_2^{-(0)}=u_2^{+(0)},\quad \sigma_{23}^{-(0)}=\sigma_{23}^{+(0)},\quad \xi_3^-=1. \tag{44}$$

As above, quantity $\sigma_{23}^{-(0)}$ is expressed as (34). Then, (43)$_3$ and (44)$_1$ imply:

$$u_2^{-(0)}=u_2^{+(0)}\Big|_{\xi_3^-=1}. \tag{45}$$

Finally, (43)$_2$ yields:

$$\sigma_{12}^{-(0)}=\frac{\partial u_2^{+(0)}}{\partial \xi_1}\Bigg|_{\xi_3^-=1}. \tag{46}$$

Hence, shear displacement and stress depend only on the deformation of the substrate.

4.3. Stiff layer, $\alpha \geq 1$

For a stiff layer, we scale the displacements and stresses according to the third column in Table 1:

$$u_2^-=h\varepsilon^{-2}u_2^{*-},\quad \sigma_{12}^-=\mu^-\varepsilon^{-1}\sigma_{12}^{*-},\quad \sigma_{23}^-=\mu^-\sigma_{23}^{*-}, \tag{47}$$

which implies:

$$\sigma_{12,1}^{*-}+\sigma_{23,3}^{*-}=0,\quad \sigma_{12}^{*-}=u_{2,1}^{*-},\quad \varepsilon^2\sigma_{23}^{*-}=u_{2,3}^{*-}. \tag{48}$$

Scaling for the half-space is taken as:

$$u_2^+=h\varepsilon^{-2}u_2^{*+}, \tag{49}$$

$$\sigma_{12}^+=\mu^-\varepsilon^{\alpha-1}\sigma_{12}^{*+},\quad \sigma_{23}^+=\mu^-\varepsilon^{\alpha-1}\sigma_{23}^{*+}, \tag{50}$$

which, substituted into (7), gives (25). Boundary condition, again, is expressed as (27), whereas the continuity conditions are:

$$u_2^{*-}=u_2^{*+},\quad \sigma_{23}^{*-}=\varepsilon^{\alpha-1}\sigma_{23}^{*+},\quad \xi_3^-=1. \tag{51}$$

At leading order for the layer we have:

$$\sigma_{12,1}^{-(0)}+\sigma_{23,3}^{-(0)}=0,\quad \sigma_{12}^{-(0)}=u_{2,1}^{-(0)},\quad u_{2,3}^{-(0)}=0, \tag{52}$$

with boundary condition (31). Taking into account (51)$_2$, i.e. $\sigma_{23}^{*-}\gg\sigma_{23}^{*+}$ at $\alpha>1$, and $\sigma_{23}^{*-}\sim\sigma_{23}^{*+}$ at $\alpha=1$, the continuity conditions at $\xi_3^-=1$ are:

$$u_2^{-(0)}=u_2^{+(0)}, \tag{53}$$

$$\sigma_{23}^{-(0)}=0,\quad \alpha>1,\quad \sigma_{23}^{-(0)}=\sigma_{23}^{+(0)},\quad \alpha=1. \tag{54}$$

From (52)$_3$ we have:

$$u_2^{-(0)}=V, \tag{55}$$

where: $V=V(\xi_1)$, i.e. displacement $u_2^{-(0)}$ is constant across the thickness of the layer, giving the function V, which may be denoted as a shear of the coating. Next, we deduce from (52)$_2$:

$$\sigma_{12}^{-(0)}=\frac{\partial V}{\partial \xi_1}. \tag{56}$$

Using (52)$_1$ and satisfying boundary condition (31), we obtain:

$$\sigma_{23}^{-(0)}=-\frac{\partial^2 V}{\partial \xi_1^2}\xi_3^--p^*. \tag{57}$$

Finally, from continuity conditions (54), we have:

$$\frac{\partial^2 V}{\partial \xi_1^2}=-p^*,\quad \alpha>1, \tag{58}$$

$$\frac{\partial^2 V}{\partial \xi_1^2}=-p^*-\sigma_{23}^{+(0)}\Big|_{\xi_3^-=1},\quad \alpha=1, \tag{59}$$

which are in fact the equations of plate shear, with the substrate reaction equal to 0 at $\alpha>1$.

4.4. Stiff layer, $0 \leq \alpha < 1$

In this case, the scaling for the layer is given by:

$$u_2^-=h\varepsilon^{-\alpha-1}u_2^{*-},\quad \sigma_{12}^-=\mu^-\varepsilon^{-\alpha}\sigma_{12}^{*-},\quad \sigma_{23}^-=\mu^-\sigma_{23}^{*-}, \tag{60}$$

with the governing equations:

$$\varepsilon^{1-\alpha}\sigma_{12,1}^{*-}+\sigma_{23,3}^{*-}=0, \tag{61}$$

$$\sigma_{12}^{*-}=u_{2,1}^{*-},\quad \varepsilon^{\alpha+1}\sigma_{23}^{*-}=u_{2,3}^{*-}. \tag{62}$$

Scaling for the half-space is:

$$u_2^+=h\varepsilon^{-\alpha-1}u_2^{*+},\quad \sigma_{12}^+=\mu^-\sigma_{12}^{*+},\quad \sigma_{23}^+=\mu^-\sigma_{23}^{*+}, \tag{63}$$

with equations (25). Boundary and continuity conditions are taken as (27) and (42).

The leading order equations and results are the same as in Subsection 4.2.

5. NUMERICAL COMPARISON OF THE ASYMPTOTIC RESULTS WITH THE EXACT SOLUTION

In this section the derived asymptotic results are tested by comparison with the exact solution of a problem for harmonic load (10) applied at the surface of the layer $x_3=0$. In doing so, we study the coefficient k introduced in (20).

For the exact solution, coefficient k follows from (21).

For the asymptotic results, in case of a soft layer, we use relations (35) and (36) for $\alpha>1$ and $\alpha=1$, respectively, and (45) for $0\leq\alpha<1$. Shear stress $\sigma_{23}^{-(0)}$ is uniform across the thickness of the layer, see (34), and may be transmitted to the interface, therefore, the value of the interfacial displacement $u_2^{+(0)}\big|_{\xi_3^-=1}$, due to continuity conditions (33) and (42)$_2$, may be found from a simpler problem for a homogeneous half-space with $\sigma_{23}^+=-P$. Its solution for harmonic load (10) is presented as Case 1 in Appendix.

In order to derive k for a hard layer, we solve plate shear equation (58) for $\alpha>1$ and (59) for $\alpha=1$. For the latter case, taking into account continuity condition (53), the deflection of the layer, see (55), may be again derived from a problem for a half-space with $\sigma_{23}^+=-P-\frac{\mu^+}{\alpha}\frac{\partial^2 u_2^+}{\partial \xi_1^2}\Big|_{\xi_3^+=0}$ (Case 2 in Appendix for harmonic load). The case $0\leq\alpha<1$ is identical to one for a soft layer.

As a result, asymptotic formulae for the coefficient k coincide with leading order exact solution presented in Table 2.

As an illustration, we plot the dimensionless coefficient

$$k_*=\frac{h}{\mu^-}k, \tag{64}$$

in Fig. 3 and 4 for a soft and a stiff layer, respectively, with

DOI 10.2478/ama-2018-0041

$\alpha = \log_\varepsilon \mu$, Poisson's ratios $\nu^- = 0.25$ and $\nu^+ = 0.3$, and $\varepsilon = 0.1$. Here, blue solid lines correspond to the exact solution, dashed and dot-dashed lines display the asymptotic approximations at $\alpha > 1$ (formula (35) in case of a soft coating and (58) for a stiff one) and $0 \le \alpha < 1$ (formula (45) valid for both soft and stiff layers), respectively, which have limited ranges of applicability. For a soft coating, case $\alpha = 1$ gives two-term approximation (36), which appears to be uniformly valid over the whole range of parameter α, and the associated curve, denoted by red dots in Fig. 3, is very close to the exact solution. As for a stiff layer, approximation at $\alpha = 1$, represented by formula (59), is a limiting case, displayed by the blue dot in Fig. 4, being valid only for this particular value of α, therefore there is no uniformly valid approximation. We can, however, match the derived approximations through

$$\tilde{k}_* = k_*^0 e^{-\frac{\alpha}{b}} + k_*^\infty \left(1 - e^{-\frac{\alpha}{b}}\right), \tag{65}$$

where: k_*^0 and k_*^∞ – dimensionless coefficients for approximations at $\alpha = 0$ and $\alpha > 1$, respectively, and b can be found using the value of k_* at $\alpha = 1$. For harmonic load (10), $b \approx 0.455$, and the related curve is plotted with red dots in Fig. 4.

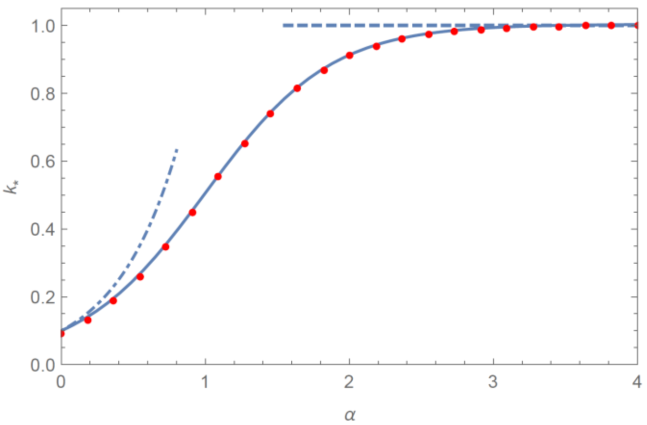

Fig. 3. Asymptotic and exact solution for harmonic load for a soft coating

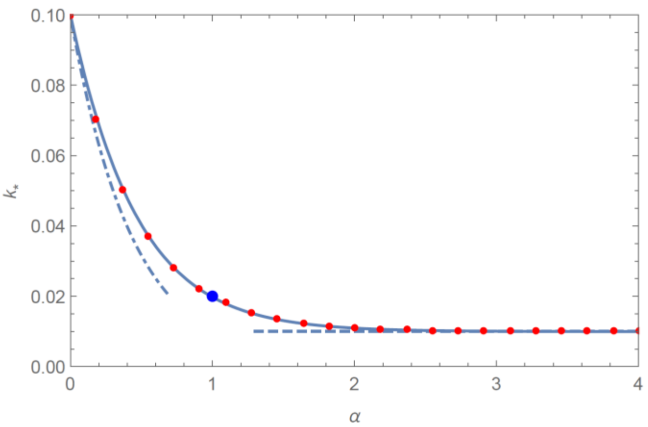

Fig. 4. Asymptotic and exact solution for harmonic load for a stiff coating

6. CONCLUDING REMARKS

A full two-parametric asymptotic analysis (in ε and μ) of the anti-plane shear deformation problem of a coated half-space is developed. It is demonstrated that in case of a relatively soft layer for a rather high contrast ($\alpha > 1$), the deformation of the substrate can be neglected, which leads to Winkler-type behaviour. In a similar situation for a relatively stiff coating ($\alpha \ge 1$), we arrive at equations of plate shear. At the same time, in the intermediate range of contrast in stiffness considered in Sections 4.2 and 4.4, the layer deformation is strongly affected by the presence of the substrate. In this case, shear deformation may be found from a simpler problem for a half-space. The latter, together with a Winkler-type behaviour term, result in two-term asymptotic formula (36) uniformly valid over the whole range of material parameter for a relatively soft layer. For a stiff coating, when the geometrical and material parameters are of the same order ($\alpha = 1$), it is also possible to reduce the original problem for a coated solid to a problem for a homogeneous half-space with effective boundary conditions at the surface.

The obtained solution is also of importance for problems of delamination between the thin coating and the substrate, especially in tribological context, see e.g. Goryacheva and Torskaya (2010), Holmberg et al. (2009) and Jiang et al. (2010).

In addition, we mention related problems for the imperfect transmission conditions, see Mishuris (2003, 2004) and Mishuris and Öchsner (2005). Note that such asymptotically evaluated simplified conditions can be verified numerically with FEM analysis, see e.g. Mishuris et al. (2005), Mishuris and Öchsner (2007). This is of crucial importance as accurate mathematical proof may not always be available. On the other hand, such conditions fail near singular points (crack tip, edges), but still may be valuable for physical applications in fracture mechanics, see Mishuris (1999, 2001).

We also note related problems on homogenization of high-contrast periodic structures, see e.g. Cherdantsev and Cherednichenko (2012), Figotin and Kuchment (1998), Kaplunov and Nobili (2017), and Smyshlyaev (2009).

APPENDIX

Consider a homogeneous elastic half-space ($\xi_3^+ \ge 0$) subject to the boundary conditions presented in Table 3.

Tab. 3. Summary for a homogeneous half-space with various cases of boundary conditions

	Case 1	Case 2
	Boundary conditions	
σ_{23}^+	$-A\mu^- \sin\xi_1$	$-A\mu^- \sin\xi_1 - \frac{\mu^+}{a}\frac{\partial^2 u_2^+}{\partial \xi_1^2}\Big\vert_{\xi_3^+=0}$
	Coefficient in (13)	
c_3	$\frac{Aa\mu^-}{\mu^+}$	$\frac{Aa\mu^-}{2\mu^+}$
	Displacements and stresses at the surface	
u_2^+	$\frac{Aa\mu^- \sin\xi_1}{\mu^+}$	$\frac{Aa\mu^- \sin\xi_1}{2\mu^+}$
σ_{12}^+	$A\mu^- \cos\xi_1$	$\frac{A\mu^- \cos\xi_1}{2}$
σ_{23}^+	$-A\mu^- \sin\xi_1$	$-\frac{A\mu^- \sin\xi_1}{2}$

The equations of the formulated problem and the solution are given by (7) and (11) with functions $(13)_2$, where the values of the coefficient c_3, corresponding to the related case of the applied boundary conditions, are also presented in Table 3, together with the displacement and stress components at the surface $\xi_3^+ = 0$.

REFERENCES

1. **Aghalovyan L.** (2015), *Asymptotic theory of anisotropic plates and shells*, Singapore: World Scientific.
2. **Ahmad M., Nolde E., Pichugin, A.V.** (2011), Explicit asymptotic modelling of transient Love waves propagated along a thin coating, *Zeitschrift für angewandte Mathematik und Physik*, 62(1), 173-181.
3. **Alexandrov V.M.** (2010), Contact problems on soft and rigid coatings of an elastic half-plane, *Mechanics of Solids*, 45(1), 34-40.
4. **Argatov I., Mishuris G.** (2016), *Contact mechanics of articular cartilage layers*, Springer.
5. **Bose S.** (2017), *High temperature coatings*, Butterworth-Heinemann.
6. **Cherdantsev M., Cherednichenko K.D.** (2012), Two-scale Γ-convergence of integral functionals and its application to homogenisation of nonlinear high-contrast periodic composites, *Archive for Rational Mechanics and Analysis*, 2014(2), 445-478.
7. **Figotin A., Kuchment P.** (1998), Spectral properties of classical waves in high-contrast periodic media, SIAM Journal on Applied Mathematics, 58(2), 683-702.
8. **Goldenveizer A.L., Kaplunov J.D., Nolde E.V.** (1993), On Timoshenko-Reissner type theories of plates and shells, *International Journal of Solids and Structures*, 30(5), 675-694.
9. **Goryacheva I.G., Torskaya E.V.** (2010), Modeling of fatigue wear of a two-layered elastic half-space in contact with periodic system of indenters, *Wear*, 268(11-12), 1417-1422.
10. **Holmberg K., Ronkainen H., Laukkanen A., Wallin K., Hogmark S., Jacobson S., Wiklund U., Souza R.M., Ståhle P.** (2009). Residual stresses in TiN, DLC and MoS2 coated surfaces with regard to their tribological fracture behaviour. *Wear*, 267(12), 2142-2156.
11. **Horgan C.O.** (1995), Anti-plane shear deformations in linear and nonlinear solid mechanics, *SIAM review*, 37(1), 53-81.
12. **Jiang H., Browning R., Whitcomb J.D., Ito M., Shimouse M., Chang T.A., Sue H.J.** (2010). Mechanical modeling of scratch behavior of polymeric coatings on hard and soft substrates, *Tribology letters*, 37(2), 159-167.
13. **Kaplunov J., Nobili A.** (2017), Multi-parametric analysis of strongly inhomogeneous periodic waveguideswith internal cutoff frequencies, *Mathematical Methods in the Applied Sciences*, 40(9), 3381-3392.
14. **Kaplunov J., Prikazchikov D., Sergushova O.** (2016), Multi-parametric analysis of the lowest natural frequencies of strongly inhomogeneous elastic rods, *Journal of Sound and Vibration,* 366, 264-276.
15. **Kaplunov J., Prikazchikov D., Sultanova L.** (2018), Justification and refinement of Winkler-Fuss hypothesis, *Zeitschrift für Angewandte Mathematik und Physik,* 69(3), 80.
16. **Kaplunov J., Prikazchikov D.A.** (2017), Asymptotic theory for Rayleigh and Rayleigh-type waves, *Advances in Applied Mechanics*, 50, 1-106, Elsevier.
17. **Kaplunov J., Prikazchikov D.A., Prikazchikova, L.A.** (2017), Dispersion of elastic waves in a strongly inhomogeneous three-layered plate, *International Journal of Solids and Structures*, 113, 169-179.
18. **Li M., Liu Q., Jia Z., Xu X., Cheng Y., Zheng Y., Xi T., Wei S.** (2014), Graphene oxide/hydroxyapatite composite coatings fabricated by electrophoretic nanotechnology for biological applications, *Carbon*, 67, 185-197.
19. **Mishuris G.** (2001), Interface crack and nonideal interface concept (Mode III), *International Journal of Fracture*, 107(3), 279–296.
20. **Mishuris G.** (2003), Mode III interface crack lying at thin nonhomogeneous anisotropic interface. Asymptotics near the crack tip, *IUTAM Symposium on Asymptotics, Singularities and Homogenisation in Problems of Mechanics*, 251-260.
21. **Mishuris G.** (2004), Imperfect transmission conditions for a thin weakly compressible interface. 2D problems, *Archives of Mechanics*, 56(2), 103-115.
22. **Mishuris G. and Öchsner A.** (2005), Transmission conditions for a soft elasto-plastic interphase between two elastic materials. Plane strain state, *Archives of Mechanics*, 57(2-3), 157-169.
23. **Mishuris G. and Öchsner A.** (2007), 2D modelling of a thin elasto-plastic interphase between two different materials: plane strain case, *Composite Structures*, 80(3), 361-372.
24. **Mishuris G., Öchsner A. and Kuhn G.** (2005), FEM-Analysis of nonclassical transmission conditions between elastic structures part 1: soft imperfect interface, *CMC-TECH SCIENCE PRESS*, 2(4), 227.
25. **Mishuris G.S.** (1999), Stress singularity at a crack tip for various intermediate zones in bimaterial structures (Mode III), *International Journal of Solids and Structures*, 36(7), 999-1015.
26. **Pawlowski L.** (2008), *The science and engineering of thermal spray coatings*, John Wiley & Sons.
27. **Prikazchikova L., Ece Aydın Y., Erbaş B., Kaplunov J.** (2018), Asymptotic analysis of an anti-plane dynamic problem for a three-layered strongly inhomogeneous laminate, *Mathematics and Mechanics of Solids*, 1081286518790804.
28. **Smyshlyaev V.P.** (2009), Propagation and localization of elastic waves in highly anisotropic periodic composites via two-scale homogenization, *Mechanics of Materials*, 41(4), 434-447.
29. **Yang F.** (2006), Asymptotic solution to axisymmetric indentation of a compressible elastic thin film, *Thin Solid Films*, 515(4), 2274-2283.

Acknowledgement: The study was supported by ACORN Scholarship from Keele University. Fruitful discussions with J. Kaplunov and D. A. Prikazchikov and valuable suggestions of the referees are gratefully acknowledged. The author was also supported during her visit to Russia by grant 14.Z50.31.0036 awarded to R.E. Alexeev from Nizhny Novgorod Technical University by the Department of Education and Science of the Russian Federation.